Complex Langevin study of an attractively interacting two-component Fermi gas in 1D with population imbalance

Shoichiro Tsutsui
(RIKEN Nishina Center for Accelerator-Based Science)

In collaboration with
Takahiro M. Doi (RCNP Osaka Univ.)
Hiroyuki Tajima (Kochi Univ.)

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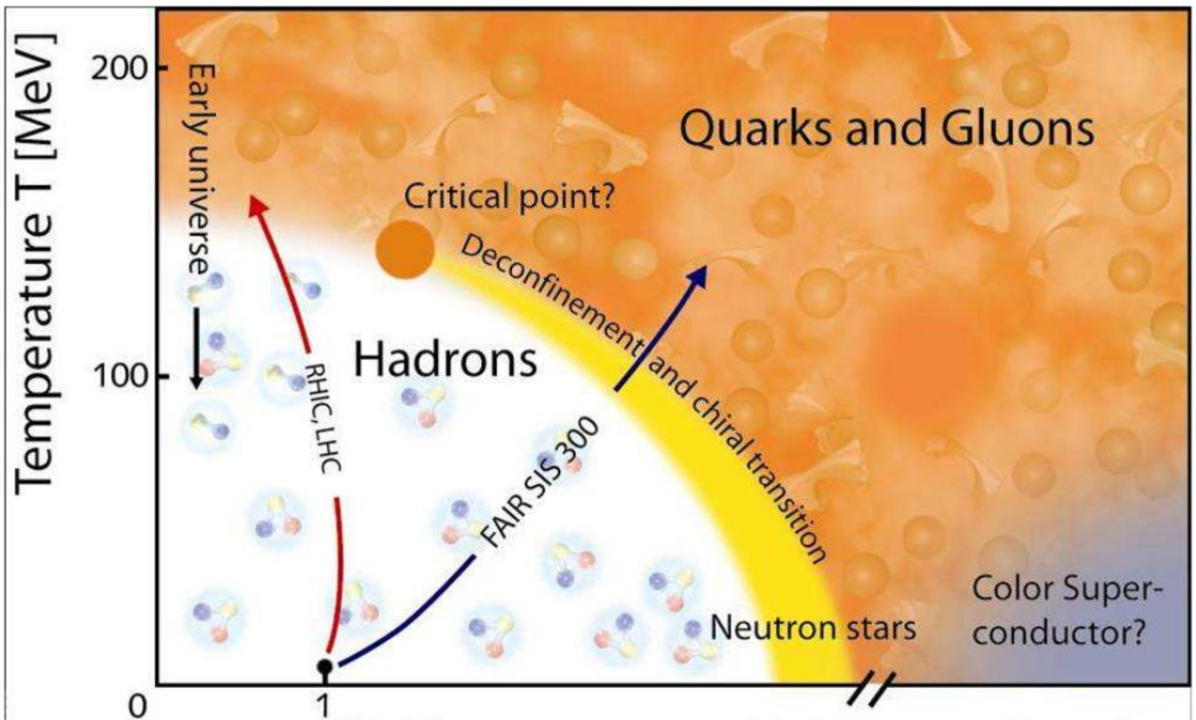
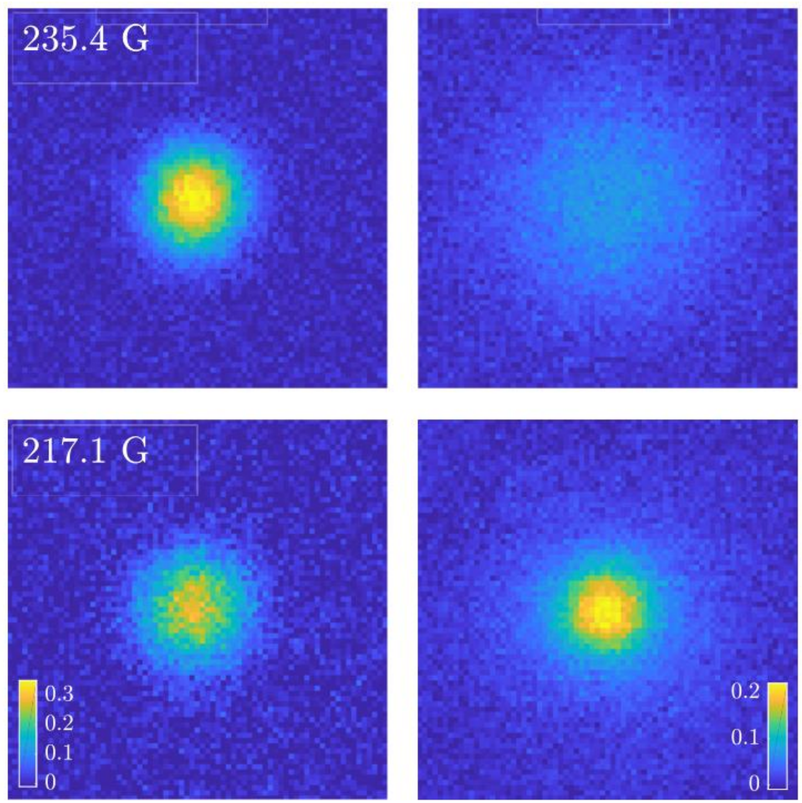
My research interest : **QCD at finite density**



J. Phys.: Conf. Ser. **706** 022004

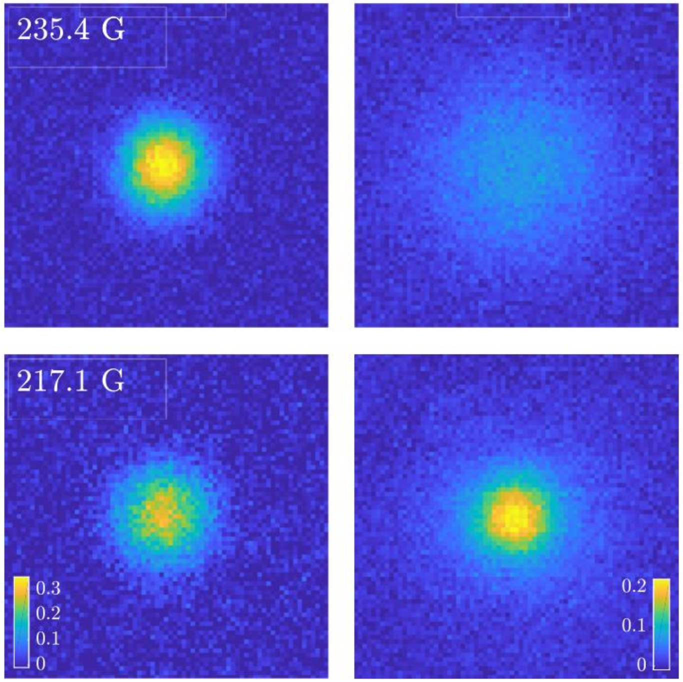
Complex Langevin study of an attractively interacting two-component **Fermi gas** in 1D with population imbalance

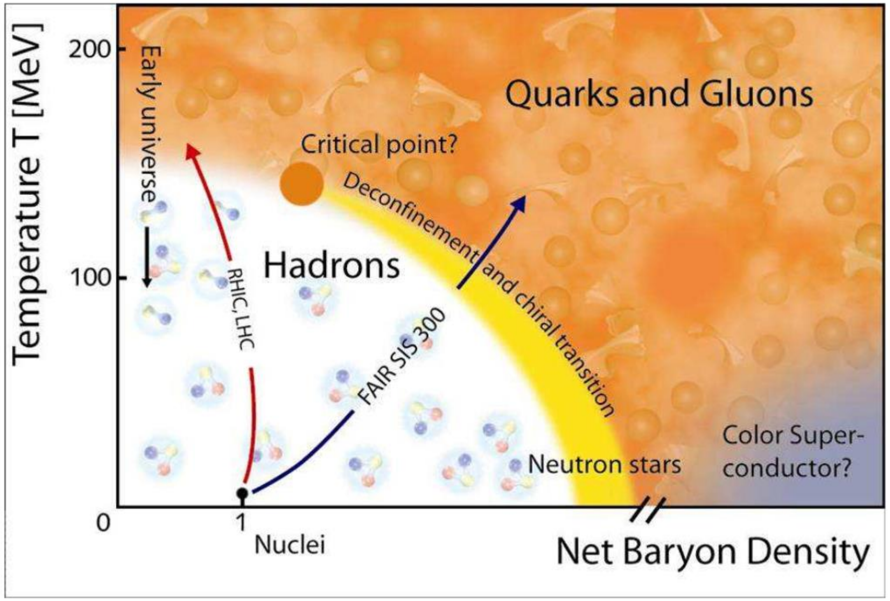
My research interest : QCD at finite density



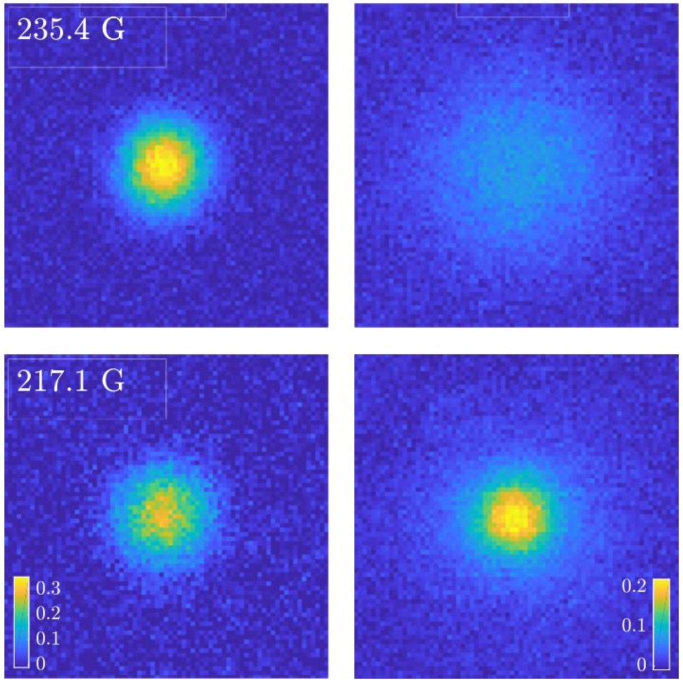
PRL **124**, 203402 (2020)

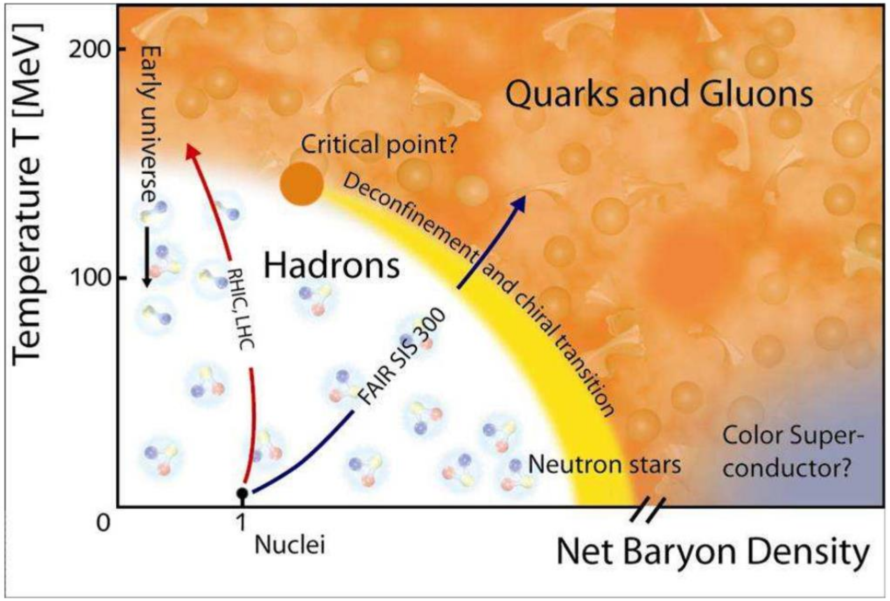
Common feature: sign problem





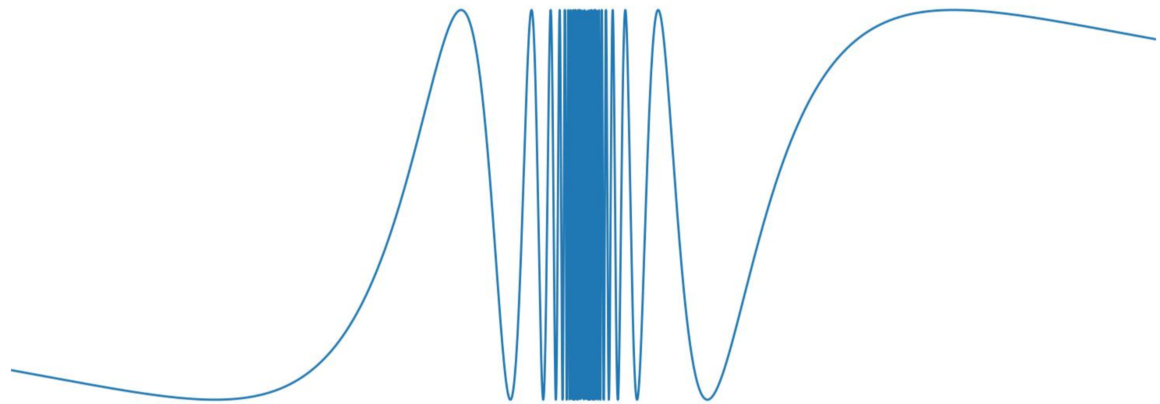
Common feature: sign problem





- What is the sign problem ?
- Sign problem in cold atom (and QCD)
- Complex Langevin (theory and application)

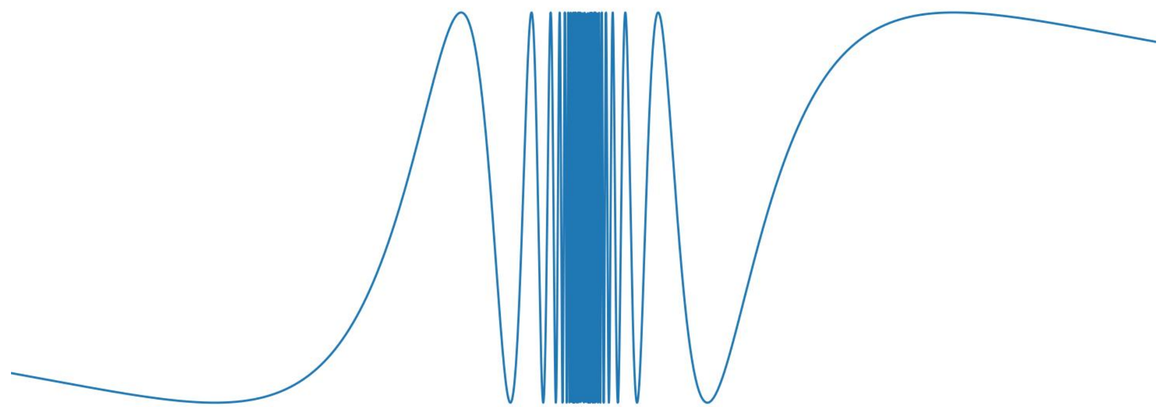
Sign problem: an intuitive picture

$\int dx \, ($  $)$

Numerical evaluation of highly oscillatory integrals is difficult

Sign problem: precise statement

$$\int dx \, \Bigg($$



$$\Bigg)$$

Monte Carlo evaluation of highly oscillatory integrals is difficult

Monte Carlo integration

Positive semi-definite

Random number

$$\int dx O(x) P(x) \sim \frac{1}{N} \sum_{i=1}^{N} O(x_i)$$

$$P(x) \propto e^{-S(x)}$$ is viewed as a probability density function if $S(x) \in \mathbb{R}$

Monte Carlo integration for complex P(x)

Non positive semi-definite

$$\frac{\int dx O(x) P(x)}{\int dx P(x)}$$

$P(x) \propto e^{-S(x)}$ is not viewed as a probability density function if $S(x) \in \mathbb{C}$

Monte Carlo integration for complex P(x)

$$\frac{\int dx O(x) P(x) / \int dx |P(x)|}{\int dx P(x) / \int dx |P(x)|}$$

Monte Carlo integration for complex P(x)

$$\frac{\int dx O(x) e^{i\theta(x)} |P(x)| / \int dx |P(x)|}{\int dx e^{i\theta(x)} |P(x)| / \int dx |P(x)|}$$

This procedure is known as reweighting.

Monte Carlo integration for complex P(x)

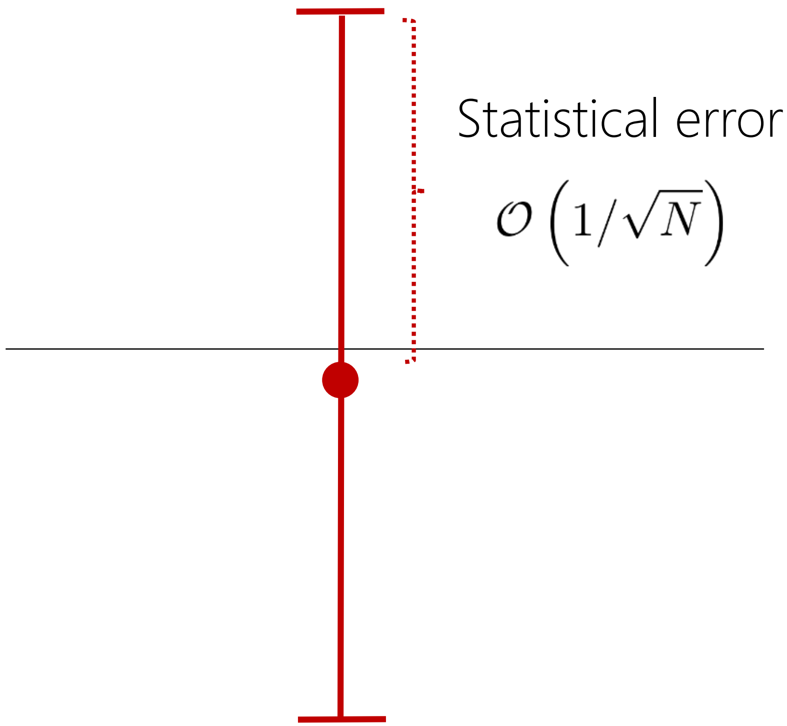
Positive semi-definite

$$\frac{\int dx O(x) e^{i\theta(x)} |P(x)| / \int dx |P(x)|}{\int dx e^{i\theta(x)} |P(x)| / \int dx |P(x)|}$$

Evaluate the numerator and denominator separately

Sign problem: more precise statement

$$\frac{\int dx e^{i\theta(x)}|P(x)|}{\int dx|P(x)|} \sim$$

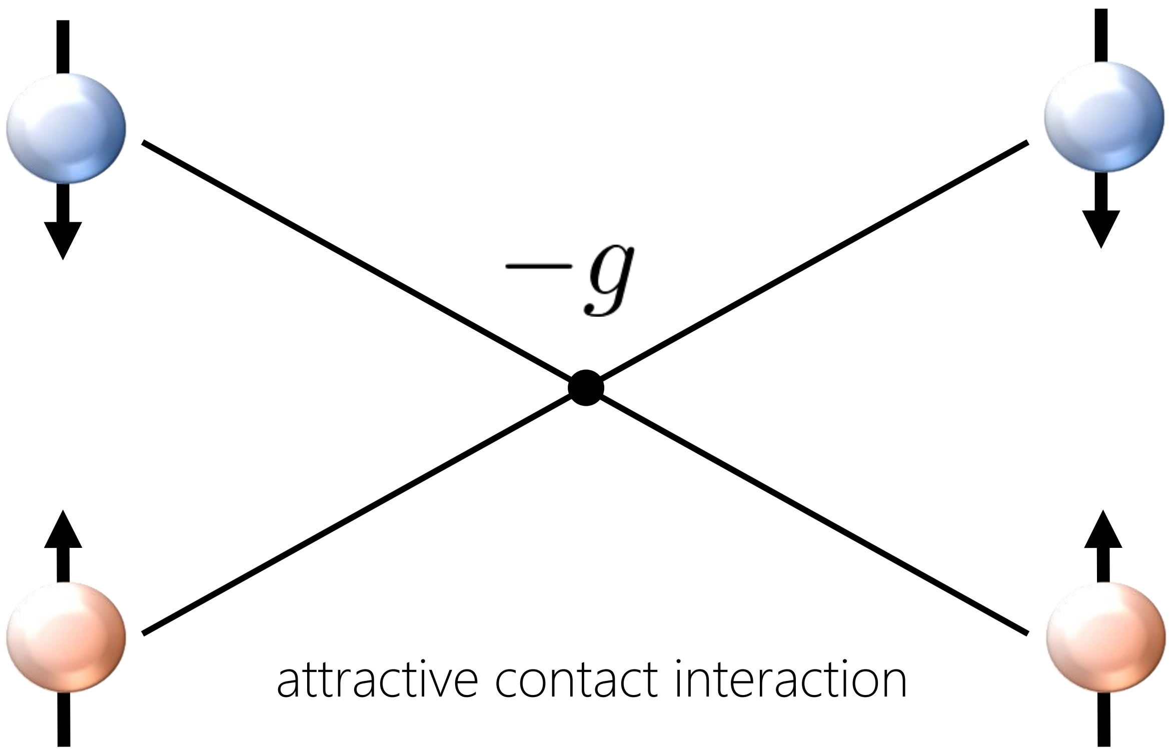


Signal-to-noise ratio is **exponentially small**

Sign problem in ultracold Fermi gas

Grand partition function

$$Z = \int \left(\prod_{\sigma} \mathcal{D}\bar{\psi}_\sigma \mathcal{D}\psi_\sigma \right) e^{-\int d\tau d^d x \left(\sum_\sigma \bar{\psi}_\sigma G_\sigma^{-1} \psi_\sigma - g \bar{\psi}_\uparrow \bar{\psi}_\downarrow \psi_\downarrow \psi_\uparrow \right)}$$

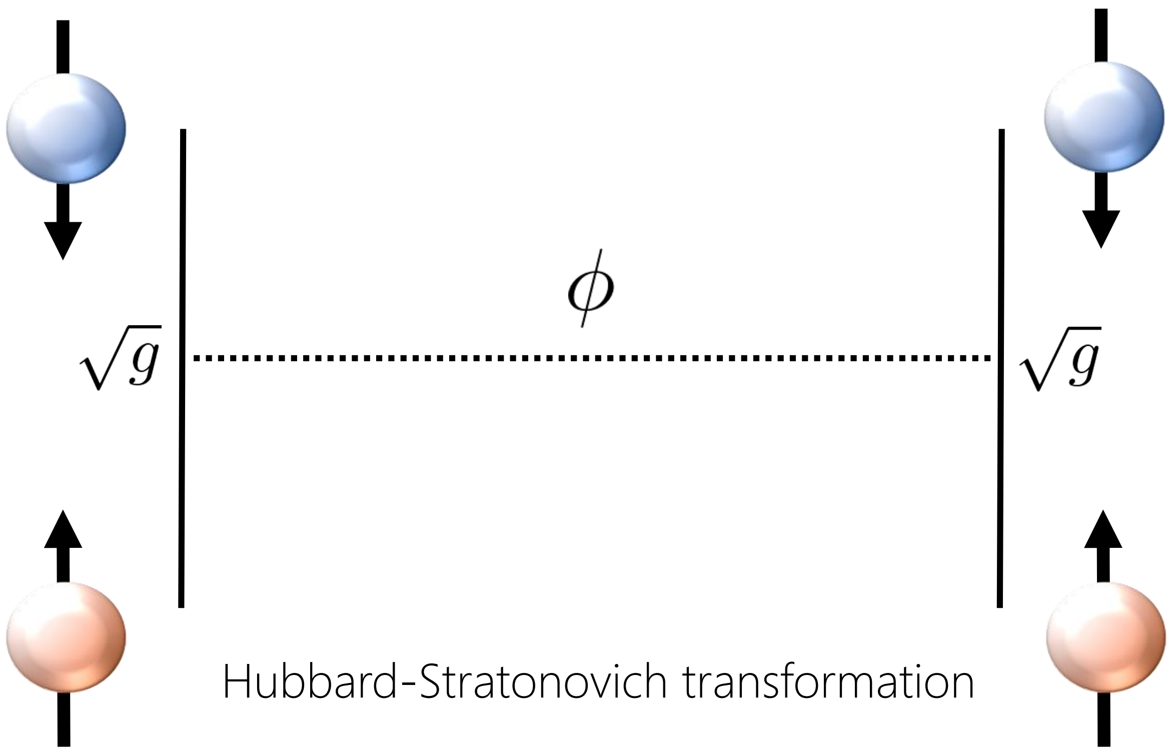


attractive contact interaction

Sign problem in ultracold Fermi gas

Grand partition function

$$Z = \int \mathcal{D}\phi \left(\prod_{\sigma} \mathcal{D}\bar{\psi}_\sigma \mathcal{D}\psi_\sigma \right) e^{-\int d\tau d^d x \left(\sum_\sigma \bar{\psi}_\sigma \left(G_\sigma^{-1} - \sqrt{g}\phi \right) \psi_\sigma + \frac{\phi^2}{2} \right)}$$



Hubbard-Stratonovich transformation

Sign problem in ultracold Fermi gas

Grand partition function

$$Z = \int \mathcal{D}\phi \det\left(G_{\uparrow}^{-1} - \sqrt{g}\phi\right) \det\left(G_{\downarrow}^{-1} - \sqrt{g}\phi\right) e^{-\int d\tau d^d x \frac{\phi^2}{2}}$$

Non positive semi-definite

↓

Reweighting

↓

Sign problem

Except for ↑=↓

$$\det\left(G_{\uparrow}^{-1} - \sqrt{g}\phi\right) \det\left(G_{\downarrow}^{-1} - \sqrt{g}\phi\right) = \det\left(G^{-1} - \sqrt{g}\phi\right)^2 \geq 0$$

Sign problem in other systems

$$Z = \int \mathcal{D}\phi \det M(\phi) e^{-S(\phi)}$$

Fermion determinant is non positive semi-definite when

- Even species of fermions with imbalance (↑≠↓)
- Odd species of fermions
- Repulsive interaction

Related topics:

polaron, FFLO, High-Tc superconductor, Effimov effect, bose-fermi mixture, …

Sign problem in QCD

$$Z = \int \mathcal{D}U \, \det(\gamma^\mu D_\mu - m - \mu\gamma^0) e^{-S(U)}$$

Fermion determinant is non positive semi-definite when

- Chemical potential is nonzero

Condition of positivity is different from that in non-rela. system

Complex Langevin

$$\frac{d\phi}{dt} = -\frac{\partial(S(\phi) - \log\det M(\phi))}{\partial\phi} + \eta$$

Parisi, Phys. Lett. 131B (1983) 393, Klauder PRA 29 (1984) 2036

Complex Langevin

$$\frac{d\phi}{dt} = -\frac{\partial S_{\text{eff}}(\phi)}{\partial \phi} + \eta$$

Drift term White noise

Parisi, Phys. Lett. 131B (1983) 393, Klauder PRA 29 (1984) 2036

Complex Langevin

$$\frac{d\phi}{dt} = -\frac{\partial S_{\mathrm{eff}}(\phi)}{\partial \phi} + \eta$$

Reach equilibrium

$$P_{\mathrm{eq}}(\phi_{\mathrm{R}}, \phi_{\mathrm{I}})$$

Parisi, Phys. Lett. 131B (1983) 393, Klauder PRA 29 (1984) 2036

Justification of complex Langevin

If $P_{\rm eq}$ or $\frac{\partial S_{\rm eff}}{\partial \phi}$ has "good" properties,

$$\int \mathcal{D}\phi_{\rm R}\mathcal{D}\phi_{\rm I} O(\phi_{\rm R}+i\phi_{\rm I})P_{\rm eq}(\phi_{\rm R},\phi_{\rm I}) = \frac{1}{Z}\int \mathcal{D}\phi O(\phi)e^{-S_{\rm eff}(\phi)}$$

Obtained by complex Langevin

Original path integral

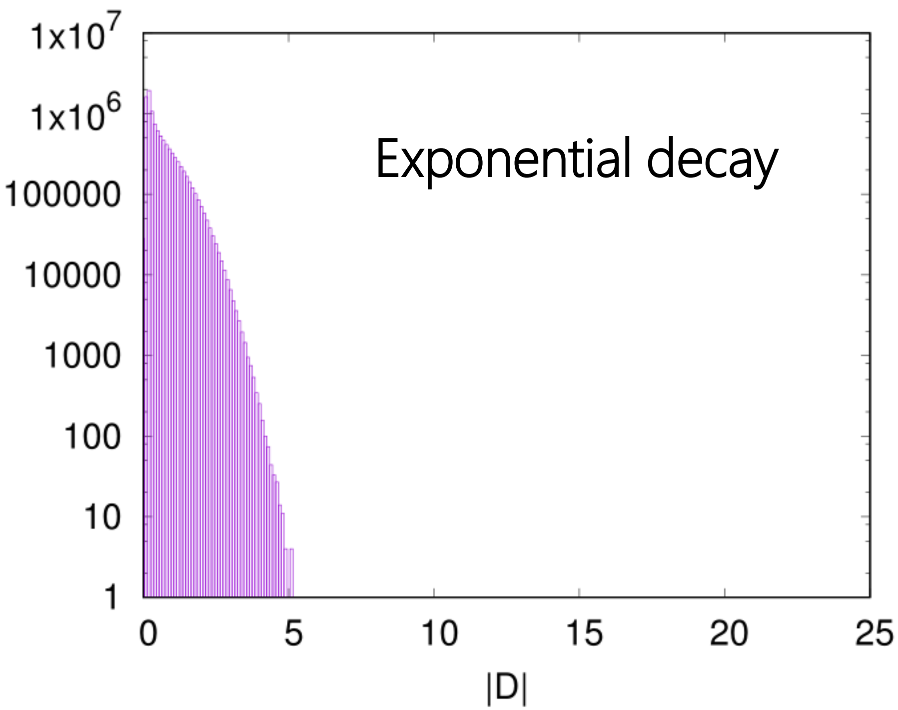
Aarts, Seiler, Stamatescu, PRD 81 (2010) 054608

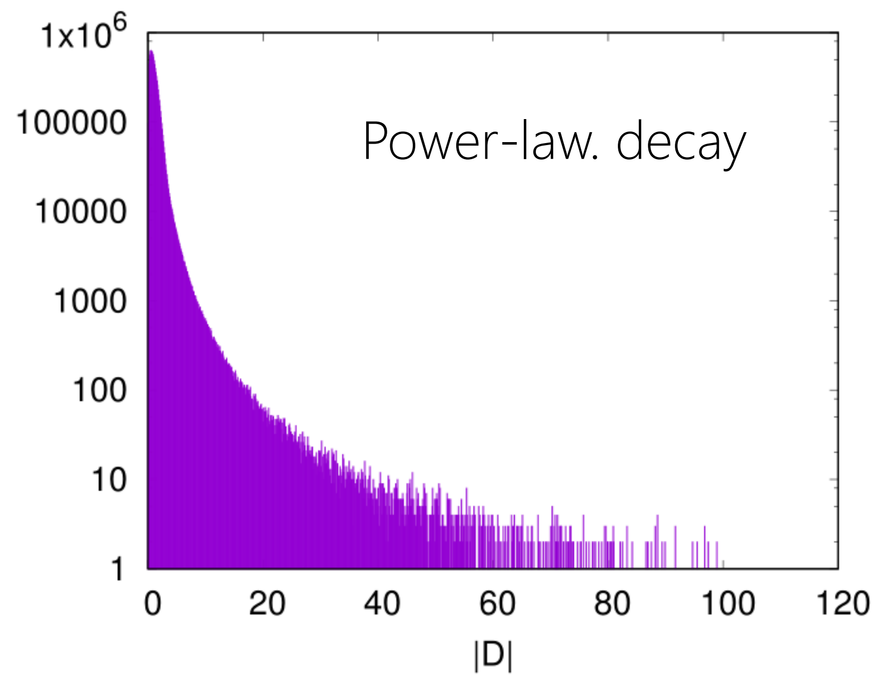
Aarts, James, Seiler, Stamatescu, EPJ C71 (2011) 1756

Nagata, Nishimura, Shimasaki, PRD 92 (2015) 011501, PTEP 2016 013B01

Practically useful criterion

Distribution of the drift term should decay **exponentially**.





Nagata, Nishimura, Shimasaki, PRD 92 (2015) 011501, PTEP 2016 013B01

Application

Our setup:

- Two-component Fermion ($\sigma = \uparrow, \downarrow$)
- Attractive contact interaction ($g > 0$)
- 1D

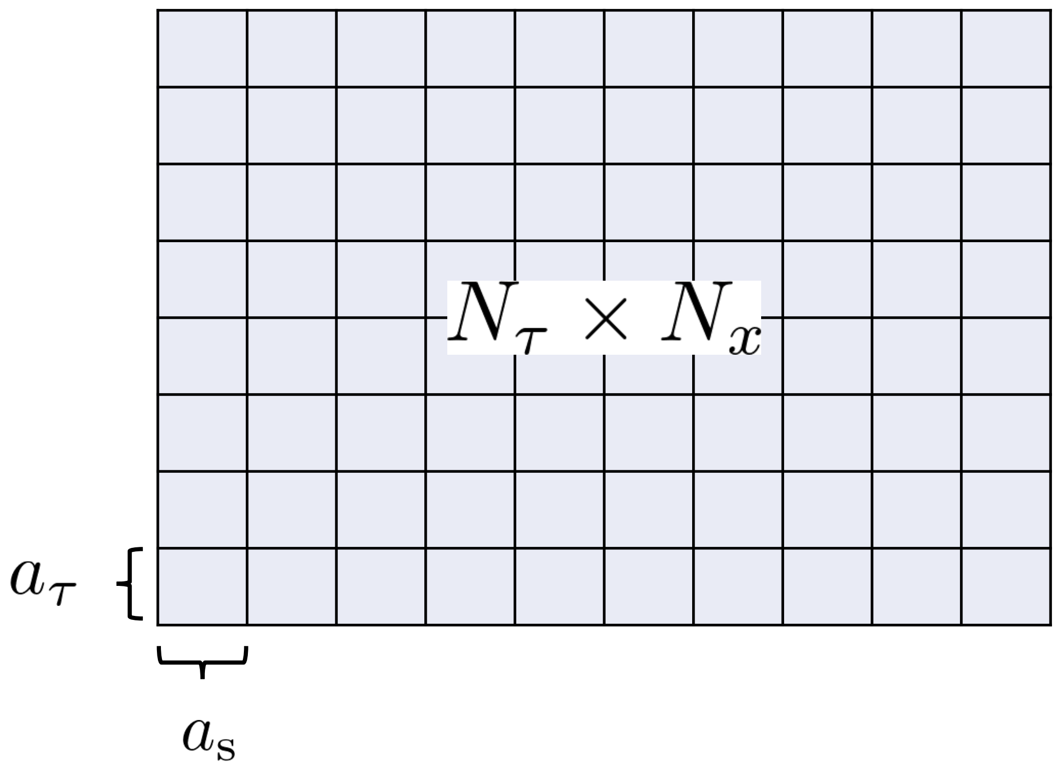
$$S = \int_0^\beta d\tau \int dx \left[\sum_{\sigma=\uparrow,\downarrow} \bar{\psi}_\sigma \left(\frac{\partial}{\partial\tau} - \frac{1}{2m_\sigma}\frac{\partial^2}{\partial x^2} - \mu_\sigma \right) \psi_\sigma - g\bar{\psi}_\uparrow\bar{\psi}_\downarrow\psi_\downarrow\psi_\uparrow \right]$$

Corresponding Hamiltonian: $\hat{H} = -\sum_{\sigma=\uparrow,\downarrow}\sum_i \frac{1}{2m_\sigma}\frac{d^2}{dx_i^2} - \sum_{i<j} g\delta(x_i - x_j)$

Application

Our setup:

- Two-component Fermion ($\sigma = \uparrow, \downarrow$)
- Attractive contact interaction ($g > 0$)
- 1D
- Lattice regularization



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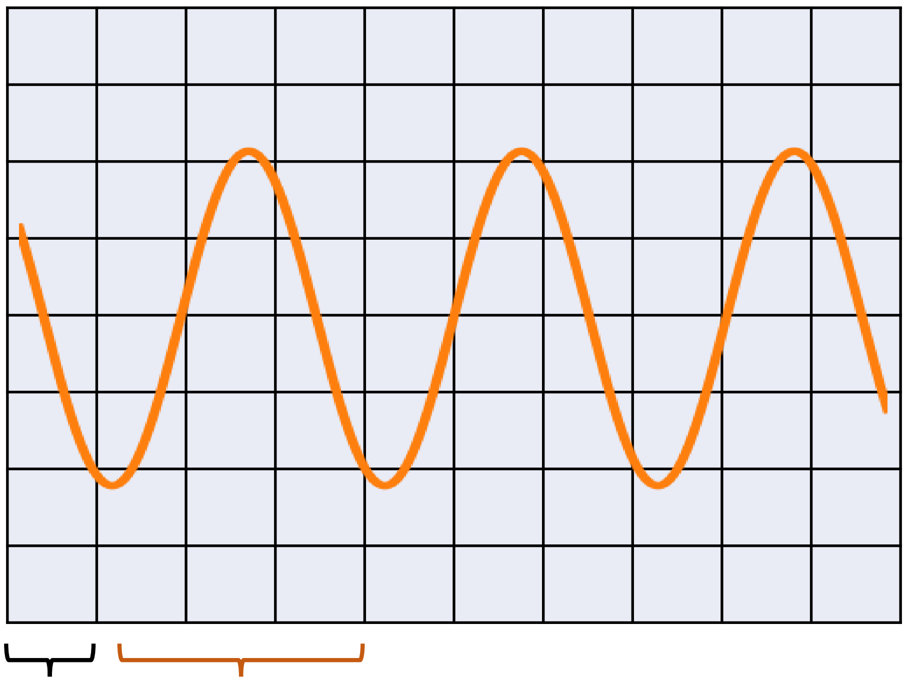

$$a_{\mathrm{s}} \ll \lambda_T = \sqrt{2\pi\beta}$$

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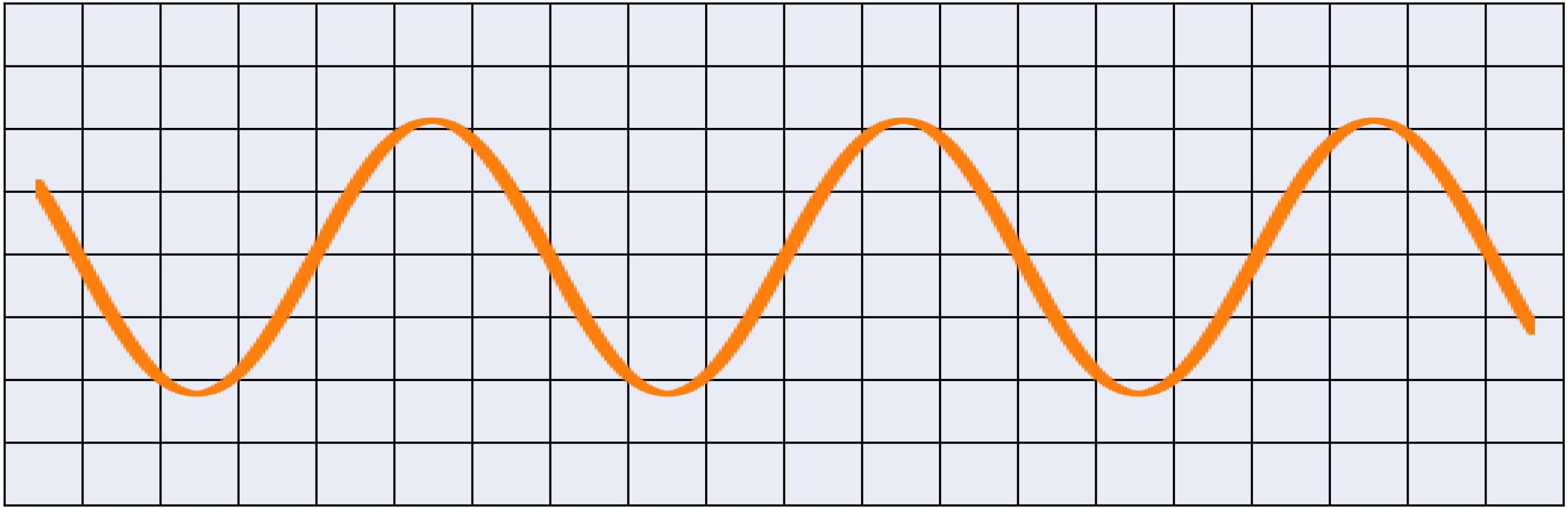


Continuum limit: $a_{\rm s} \ll \lambda_T = \sqrt{2\pi\beta} \to \infty$

Dimensionless parameters

$$\beta\mu = \beta(\mu_\uparrow + \mu_\downarrow)/2$$

$$\beta h = \beta(\mu_\uparrow - \mu_\downarrow)/2$$

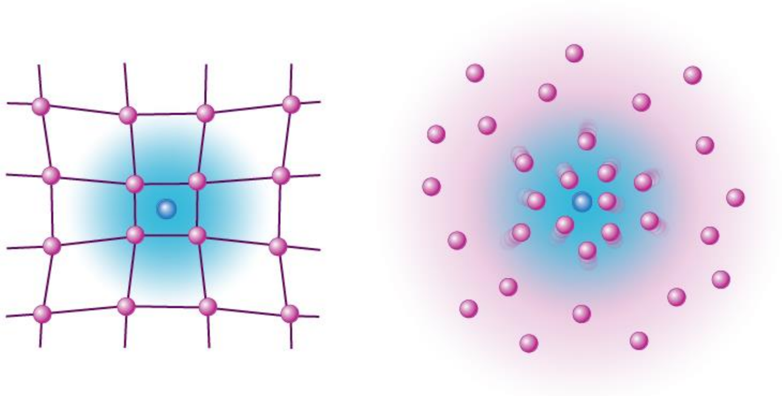
$$\lambda = \sqrt{g^2\beta}$$

$$r = a_\tau / a_{\mathrm{s}}$$

We set* $m_\uparrow = m_\downarrow = 1$

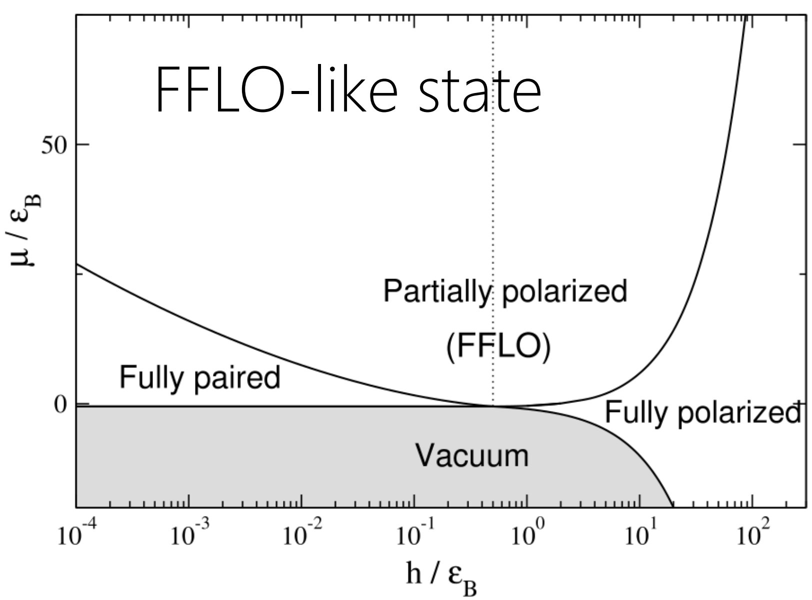
* This is not the natural unit, where c=1 !

What is expected ?



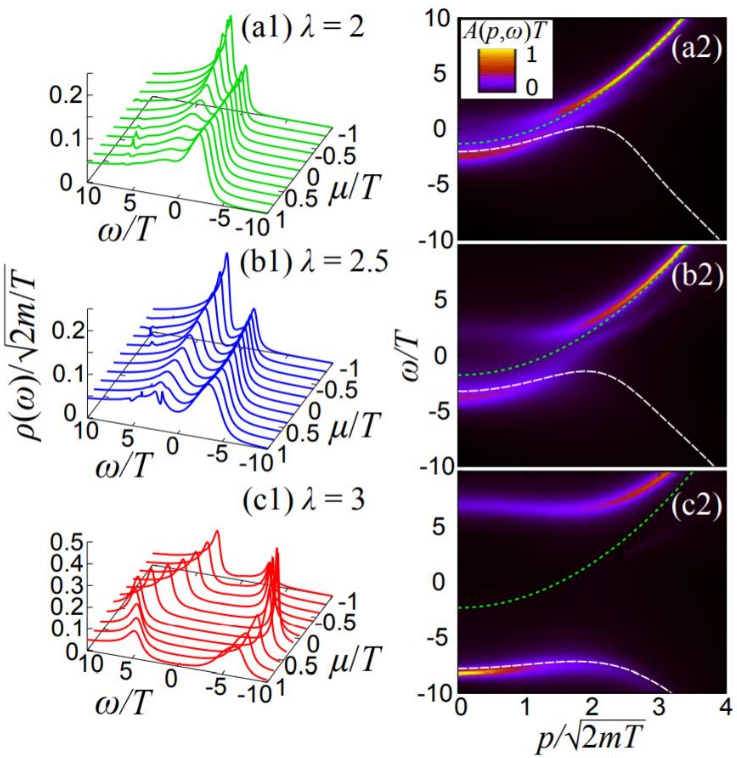
Poralon (inpurity dressed by medium)

https://physics.aps.org/articles/v9/86



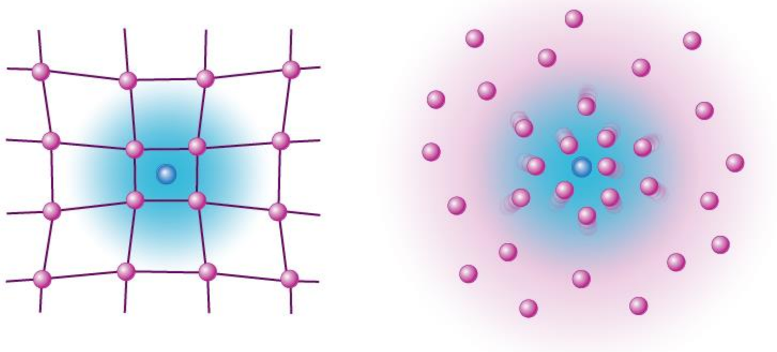
Orso, PRL 98 (2007) 070402

Pseudogap



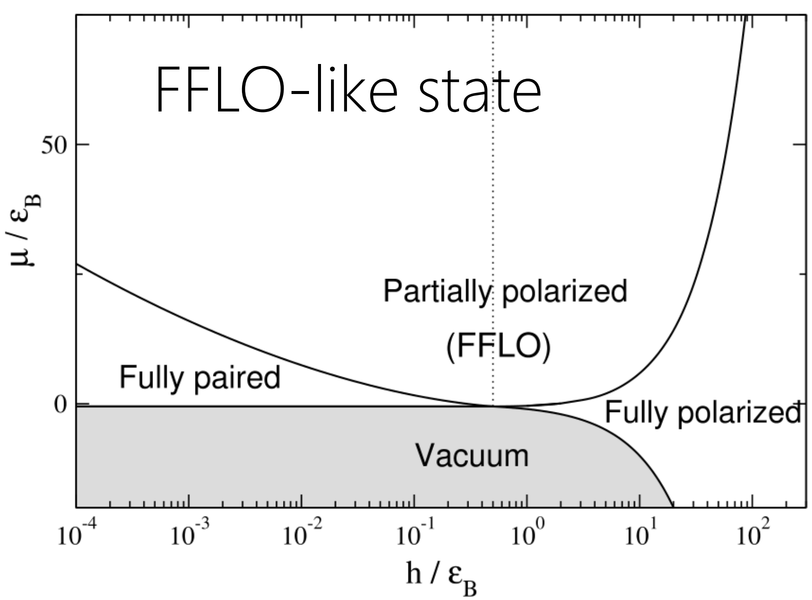
Tajima, ST, Doi, arXiv:2005.12124

What is expected ?



Poralon ← Today's topic

https://physics.aps.org/articles/v9/86



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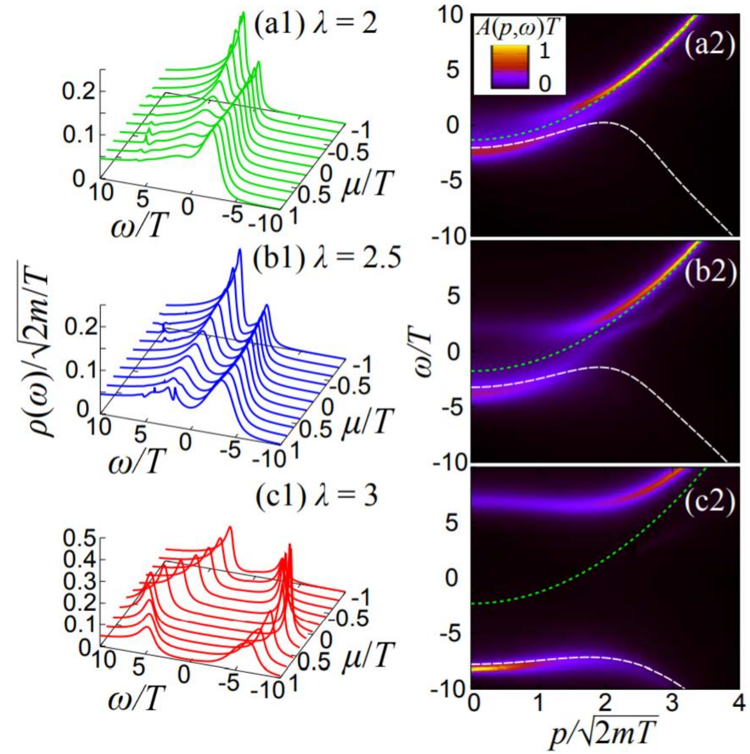
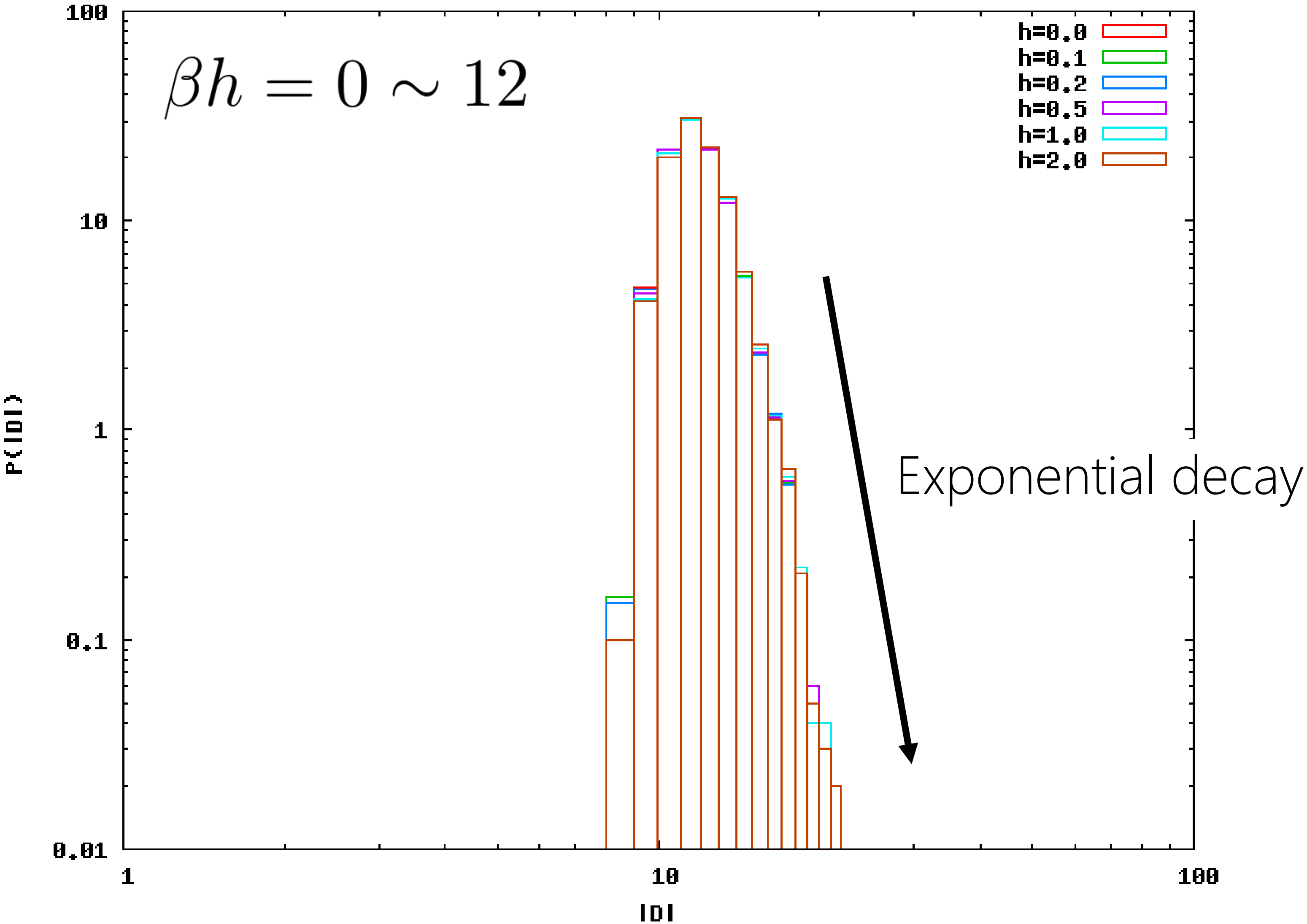


Tajima, ST, Doi, arXiv:2005.12124

Complex Langevin works !



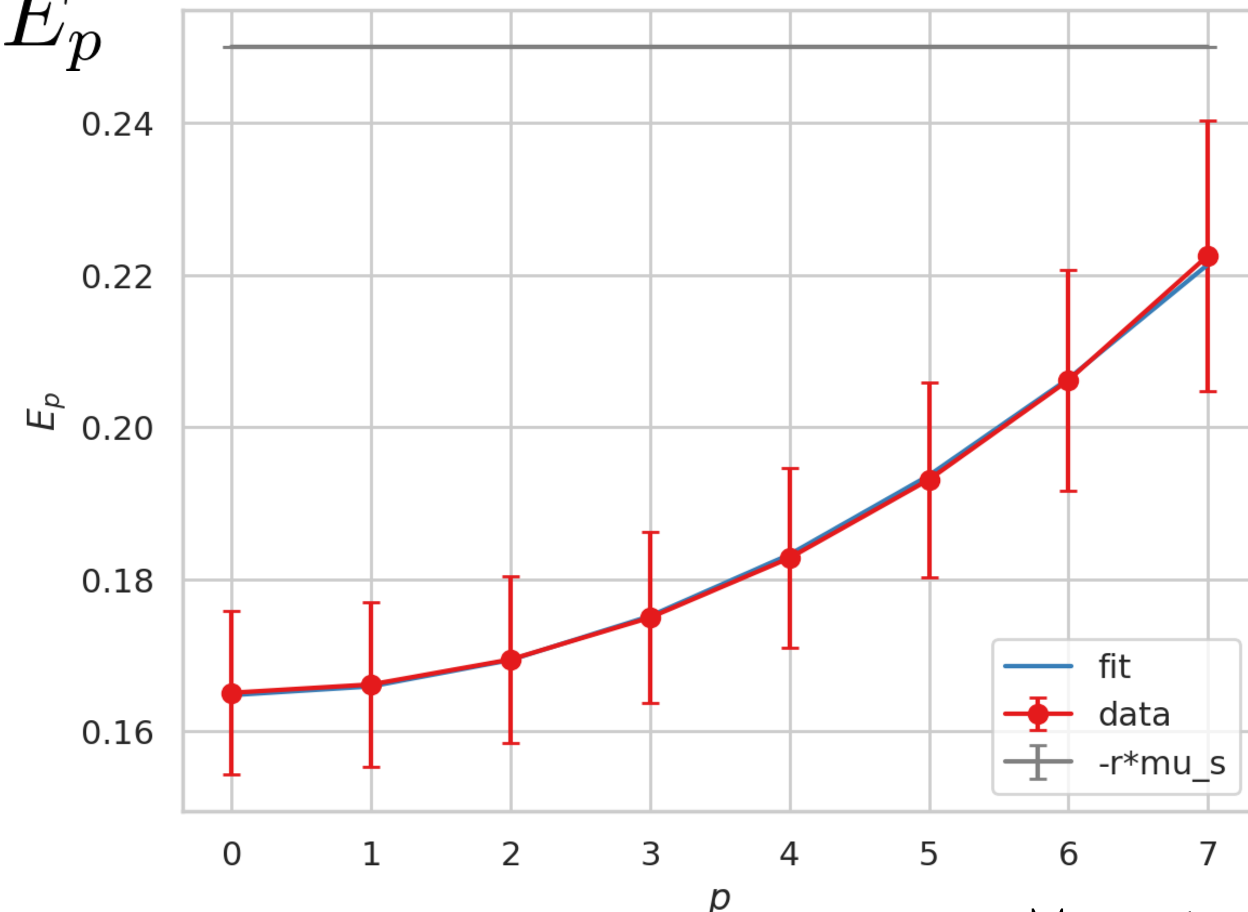
$N_x = 60, N_\tau = 40, r = 0.2, \beta\mu = 2, \lambda = 2$

Extracting the polaron energy

$$
\begin{aligned}
G(\tau) &= \langle 0|\psi_\downarrow(\tau)\psi_\downarrow{}^\dagger(0)|0\rangle \\
&= \langle 0|\mathrm{e}^{\hat{H}\tau}\psi_\downarrow(0)\mathrm{e}^{-\hat{H}\tau}\psi_\downarrow{}^\dagger(0)|0\rangle \\
&= \langle 0|\psi_\downarrow(0)\mathrm{e}^{-\hat{H}\tau}\psi_\downarrow{}^\dagger(0)|0\rangle \\
&= \sum_n \langle 0|\psi_\downarrow(0)\mathrm{e}^{-\hat{H}\tau}|n\rangle\langle n|\psi_\downarrow{}^\dagger(0)|0\rangle \\
&= \sum_n A_n \mathrm{e}^{-E_n\tau} \\
&\to A_0\mathrm{e}^{-E_0\tau}
\end{aligned}
$$

Dispersion relation of polaron

E_p

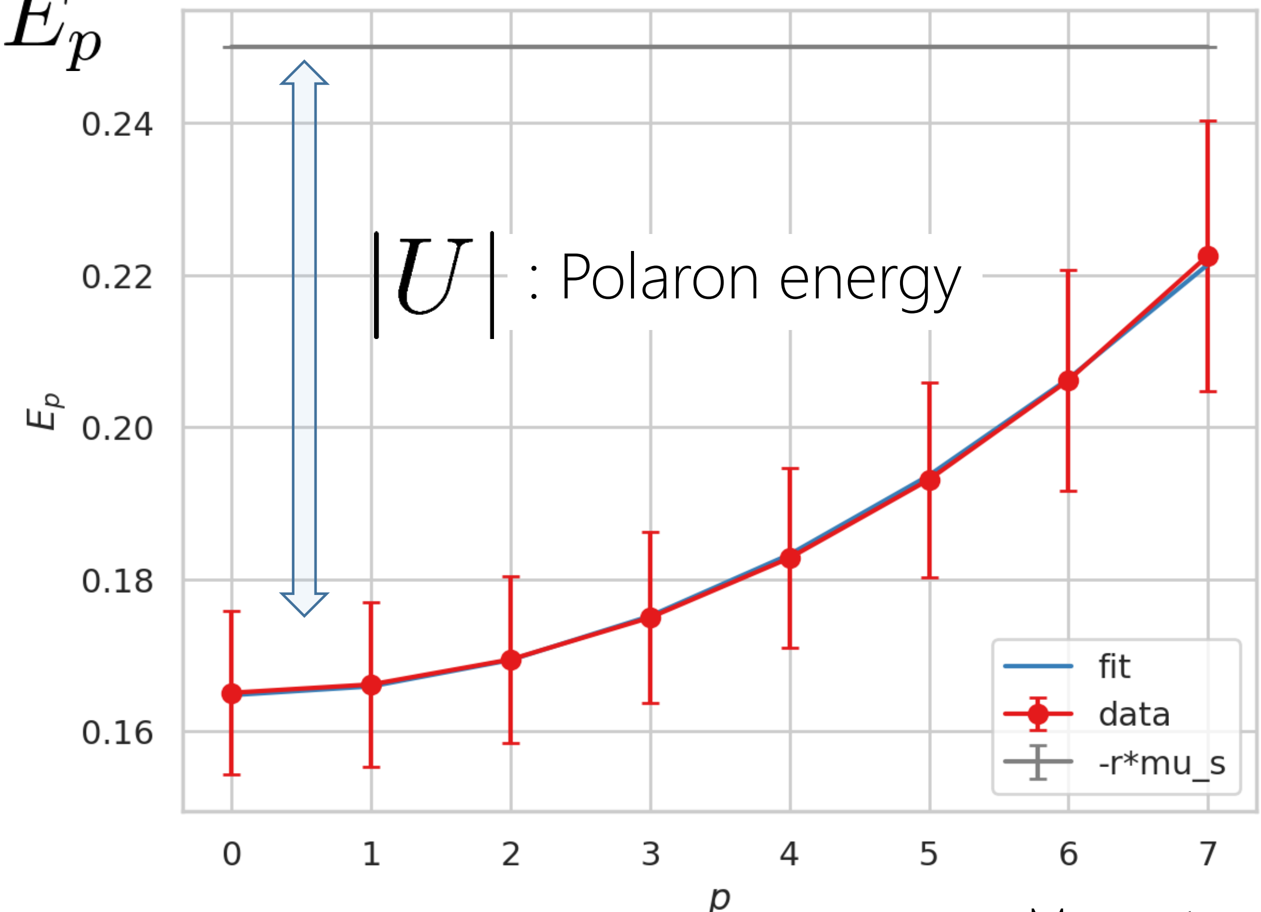


Momentum in lattice unit

Fitting function: $E_p = \dfrac{p^2}{2m_\downarrow^*} + U - r\mu_\downarrow$

Dispersion relation of polaron

E_p



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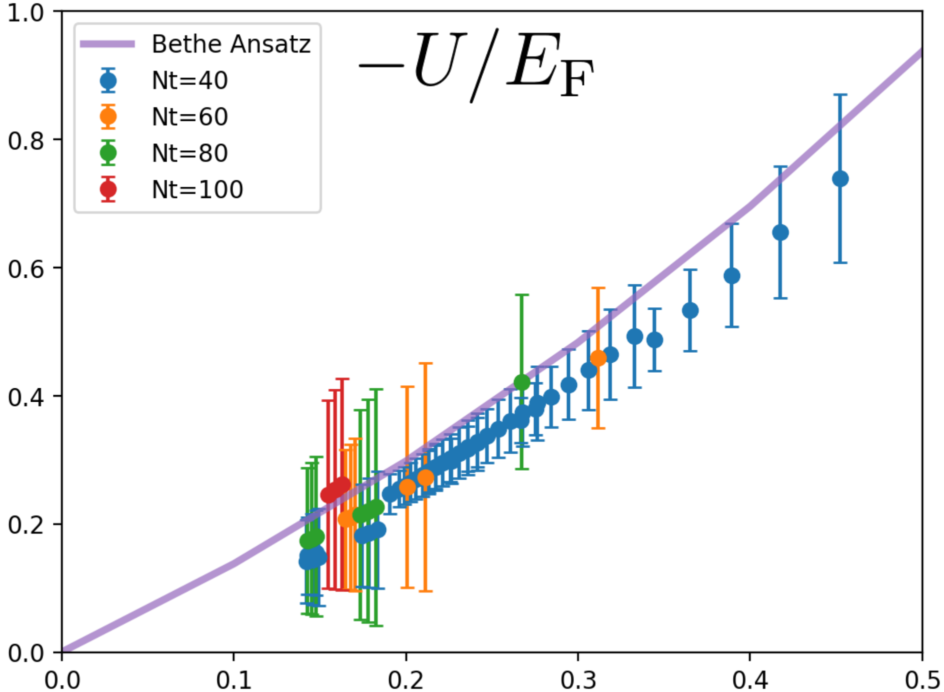
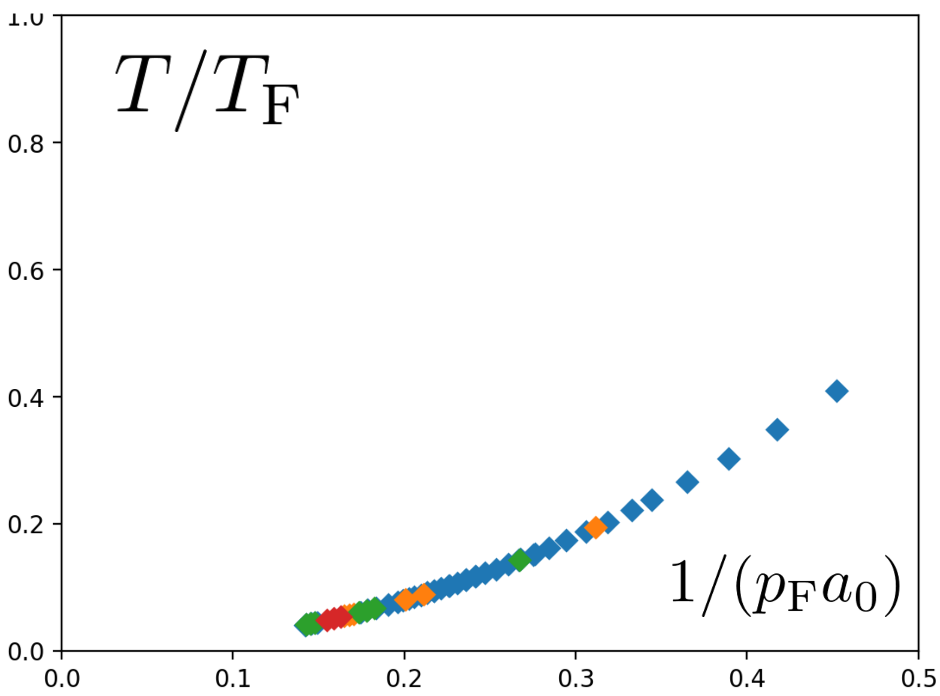


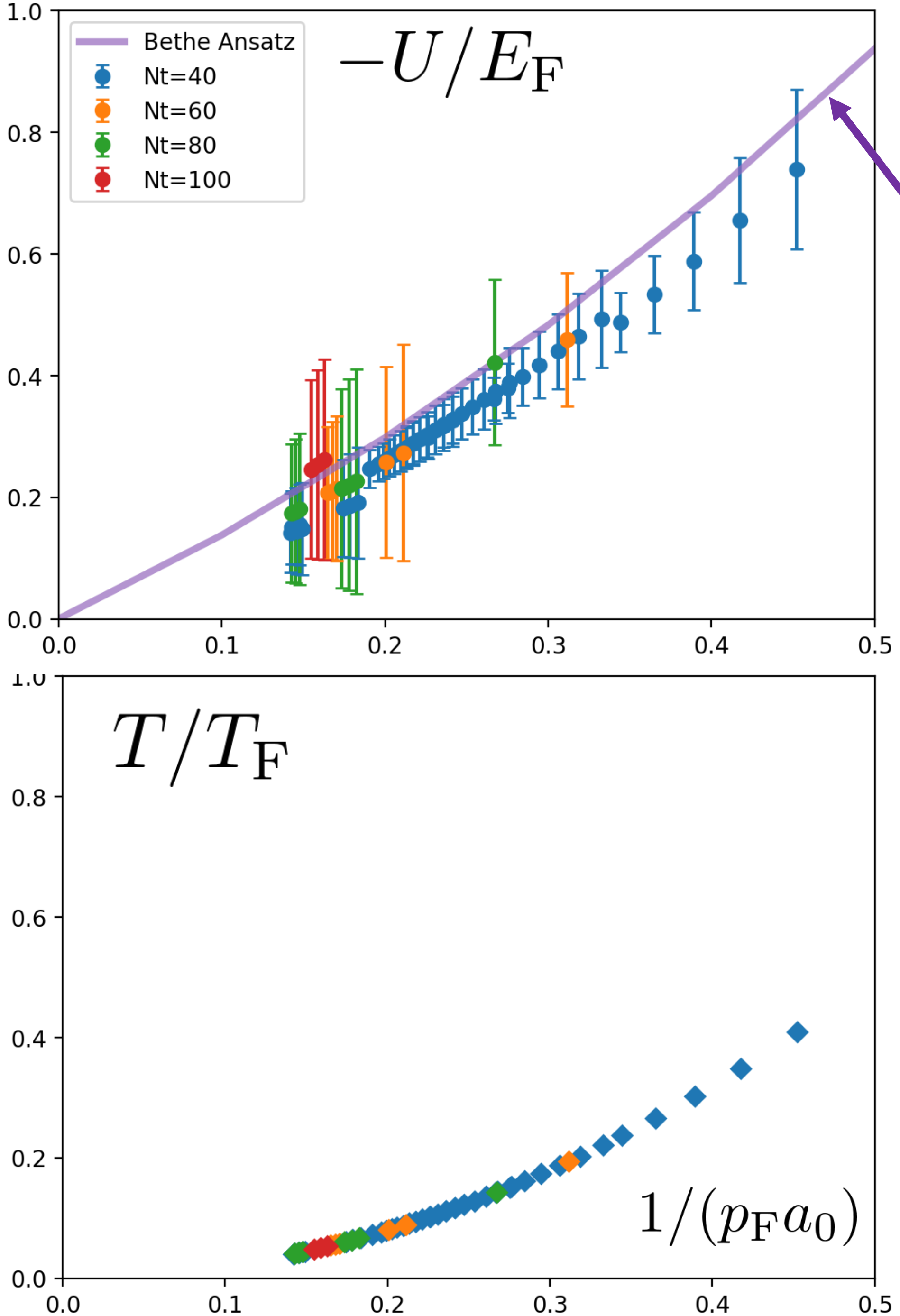
Polaron energy



Temperature

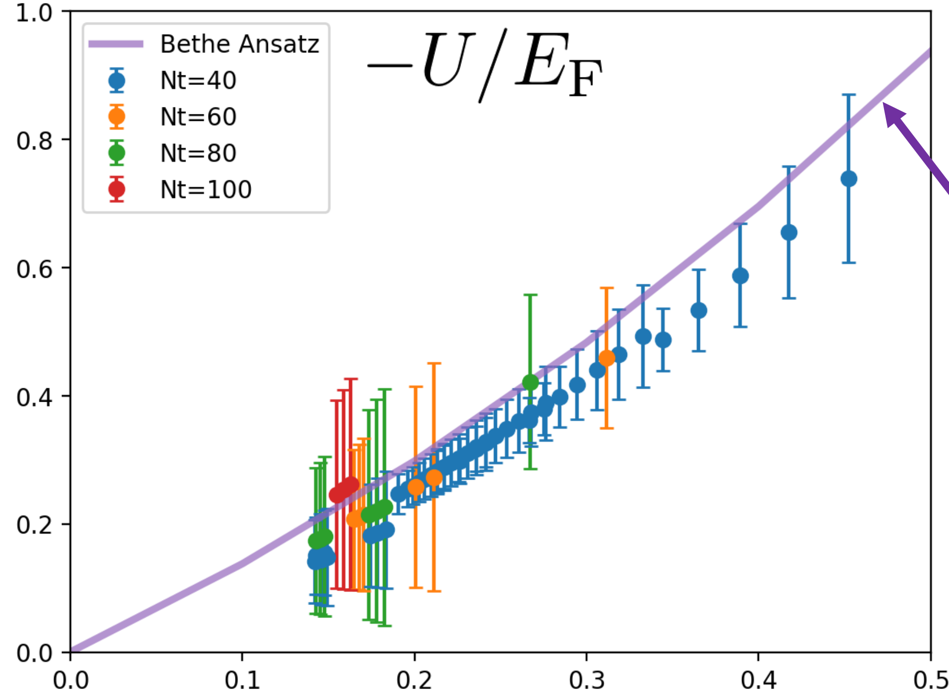
$a_0 = 2/mg$: scattering length

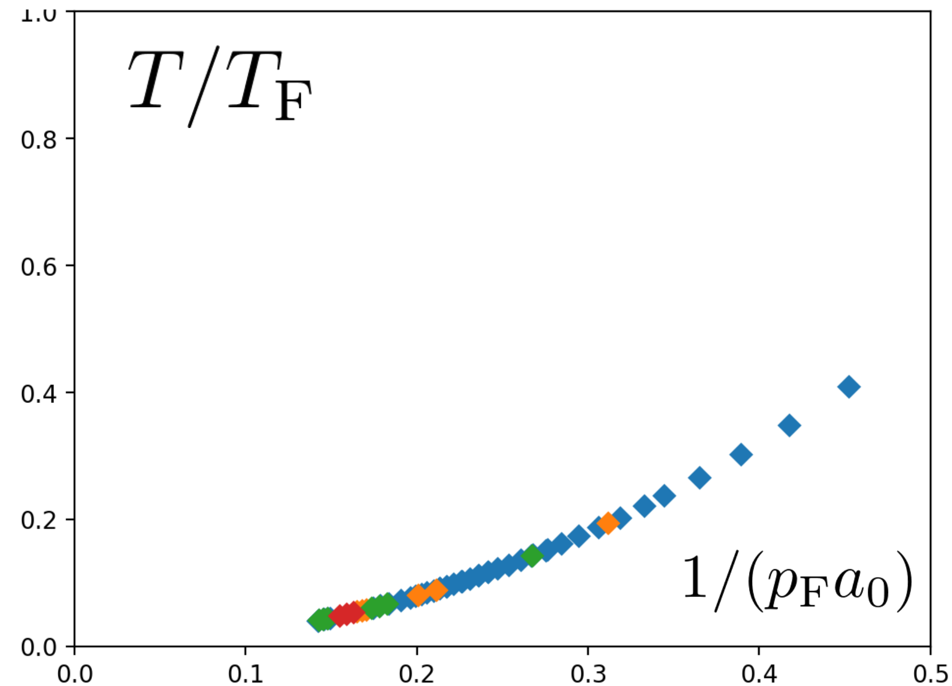
$T_{\mathrm{F}}, p_{\mathrm{F}}$: determined by $N_{\uparrow}$



Exact result at T=0 limit obtained by thermodynamic Bethe ansatz

J. B. McGuire, J. Math. Phys. 7, 123 (1966).





Exact result at T=0 limit obtained by TBA

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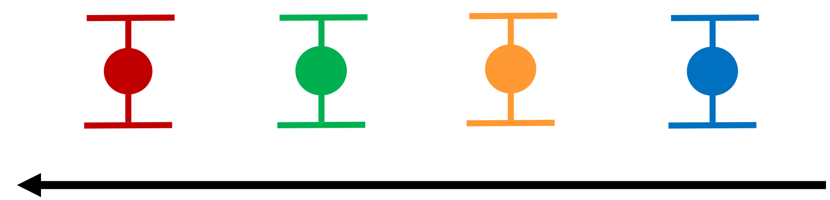
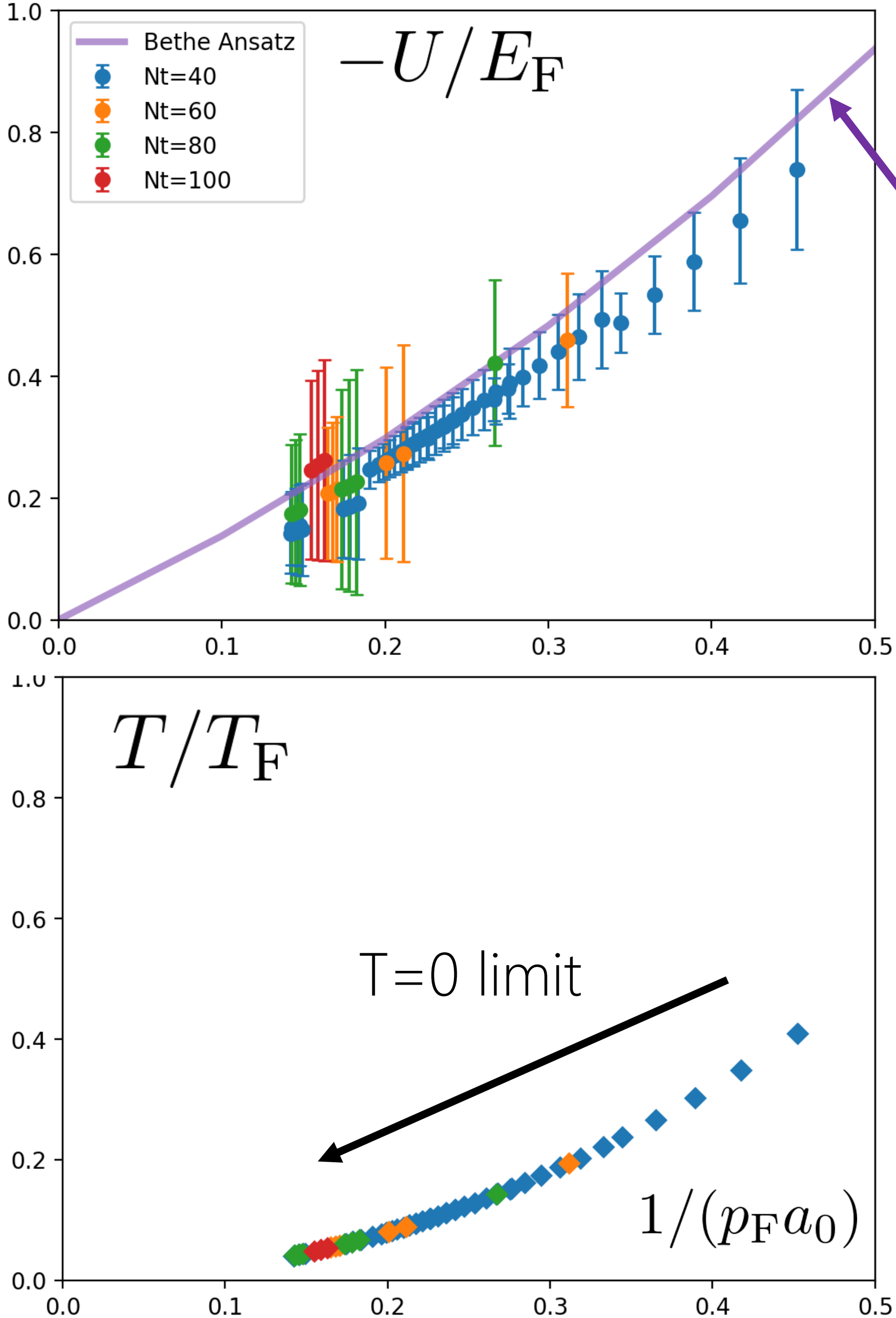


Continuum limit



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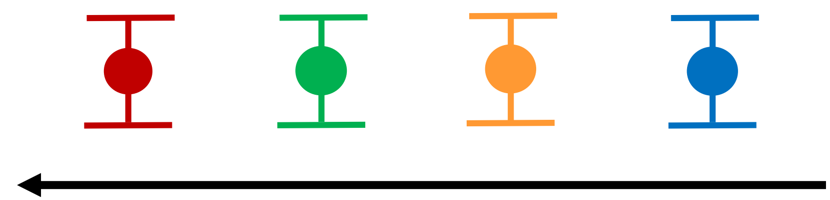

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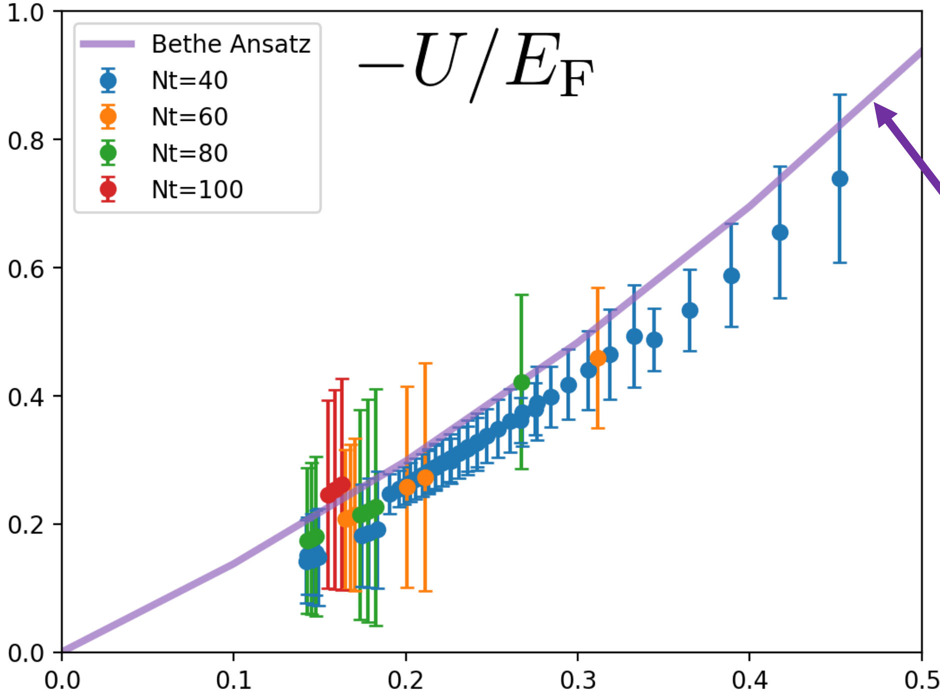

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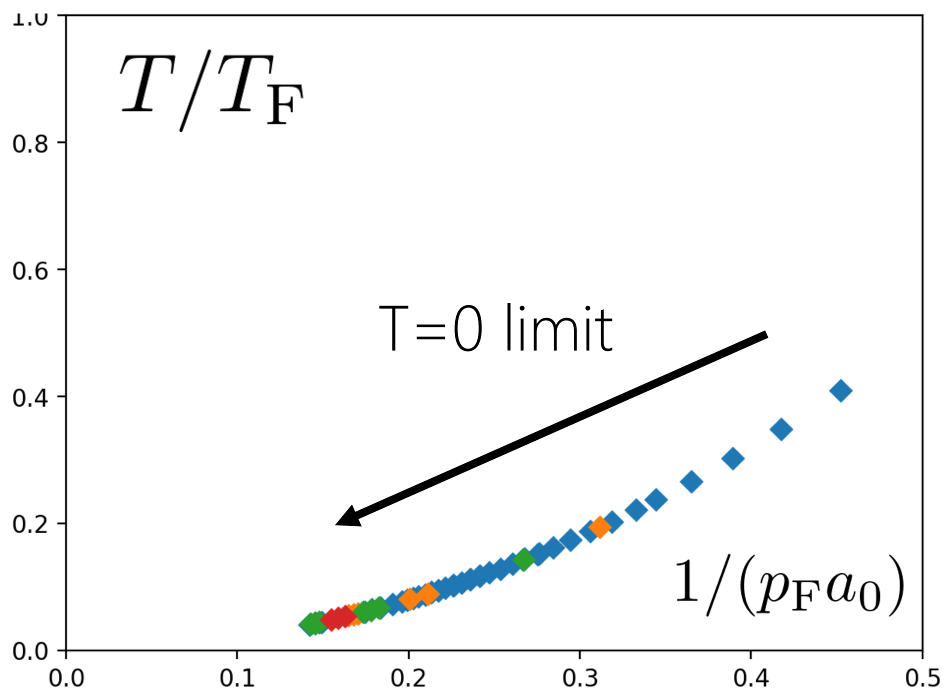
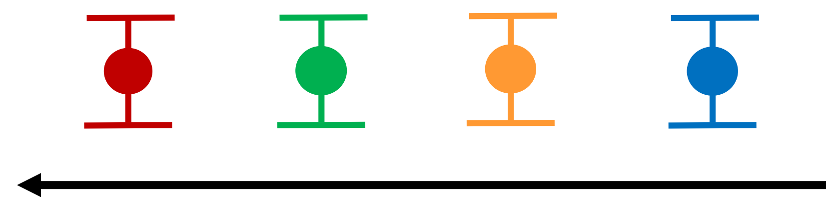




Continuum limit

Complex Langevin agrees with TBA

Summary

◆ What is the sign problem ?

- Exponentially small signal-to-noise ratio in Monte Carlo simulations

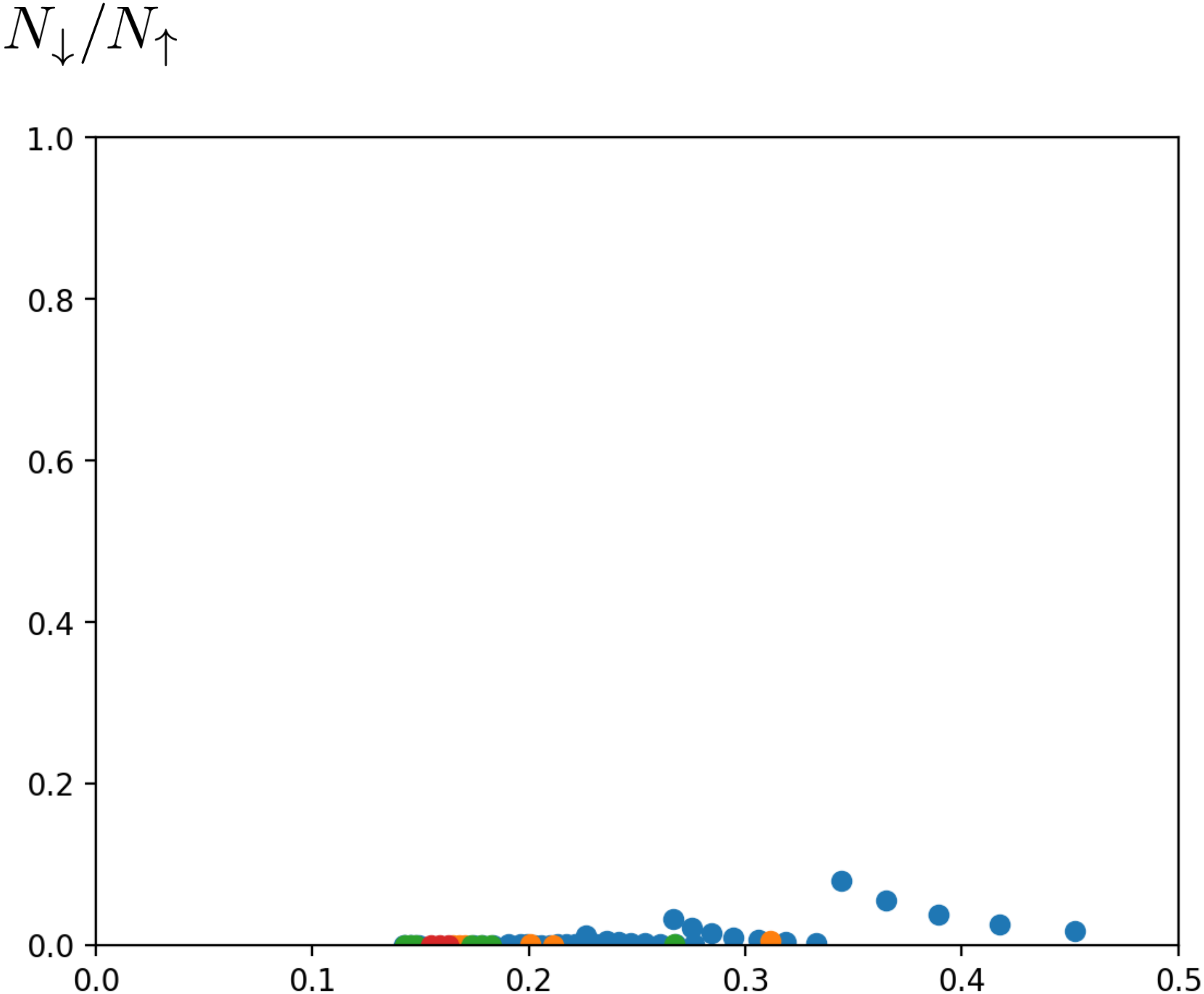
◆ Sign problem in cold atom

- Non positive definite fermion determinant causes the sign problem.

◆ Complex Langevin (theory and application)

- In our setup (1D, attractive, $\beta h \neq 0$), complex Langevin is reliable.
- We obtain polaron energy at $T \neq 0$
- Consistent with TBA

Appendix


$N_\downarrow / N_\uparrow$
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.1
0.2
0.3
0.4
0.5