Lefschetz-thimble inspired analysis of the Dykhne-Davis-Pechukas method and an application for the Schwinger Mechanism

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Summary

- Formulation of quantum tunneling

    Ours is inspired by the Lefschetz-thimble method.

    (cf. the Dykhne-Davis-Pechukas (DDP) method)

- Application to the Schwinger mechanism

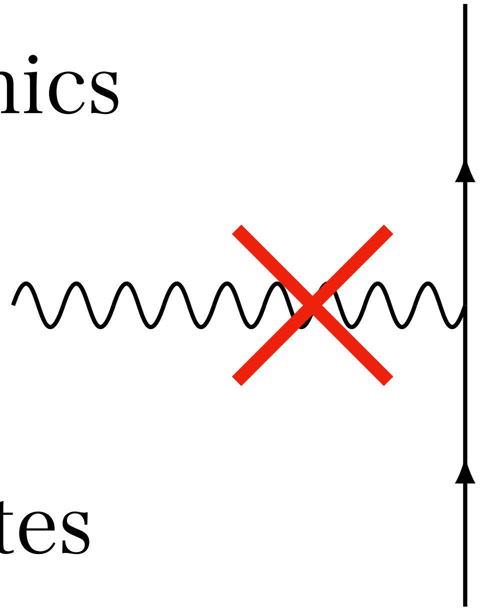
    The Schwinger mechanism can be regarded as tunneling.

Schwinger Mechanism

In an external field (anti)particle pair production makes the perturbative vacuum unstable.

- Nonperturbative effect in quantum electrodynamics

The decay width is given by $\Gamma \simeq \exp\left(-\frac{\pi m^2}{eE}\right)$.

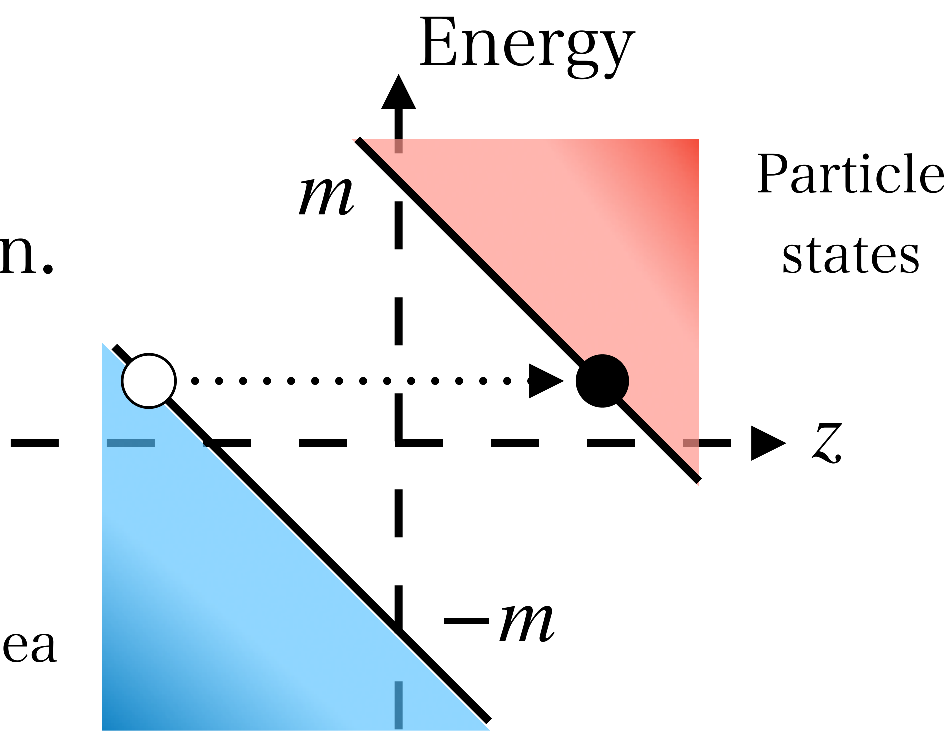


- Tunneling from antiparticle states to particle states

We consider an electric field along the z direction.

$$\boldsymbol{E} = (0\,,0\,,E_z)$$

Energies are tilted by $V(z) = -E_z z$.

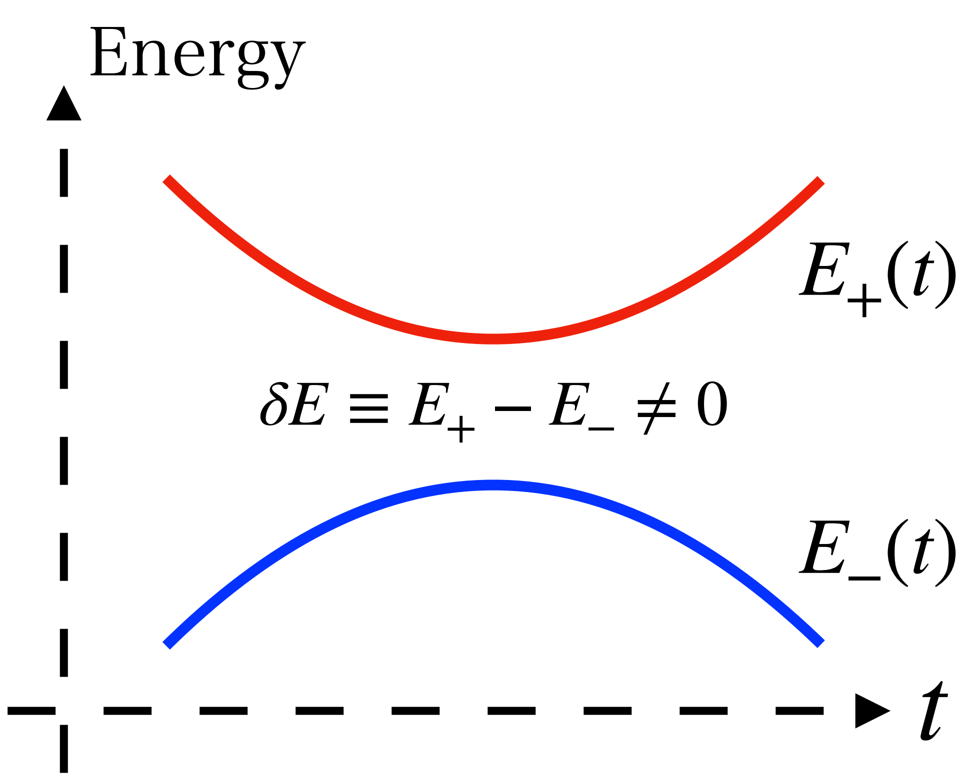


Schwinger Mechanism as Tunneling

Hamiltonian describing Schwinger mechanism is given by

$$H = \begin{pmatrix} k_z - eA(t) & m_\perp \\ m_\perp & -k_z + eA(t) \end{pmatrix} \quad \text{where} \quad \begin{cases} m_\perp^2 = m^2 + k_x^2 + k_y^2 \,, \\ E_z = -\,\partial_t A(t)\,. \end{cases}$$

Nonadiabatic energies are defined by $H(t)\psi_i(t) = E_i(t)\psi_i(t) \;\; (i = \pm\,)\,.$



For $A(t) = -\,Et\,, H$ is called the Landau-Zener model.

The transition probability is analytically calculable with an initial condition, $\psi_+(-\infty) = 0\,.$

$$P(\boldsymbol{k}) = |\,\psi_+(\infty)\,|^2 = \exp\left(-\frac{\pi m_\perp^2}{eE}\right)$$

This is the Landau-Zener formula.

Analytic Continuation

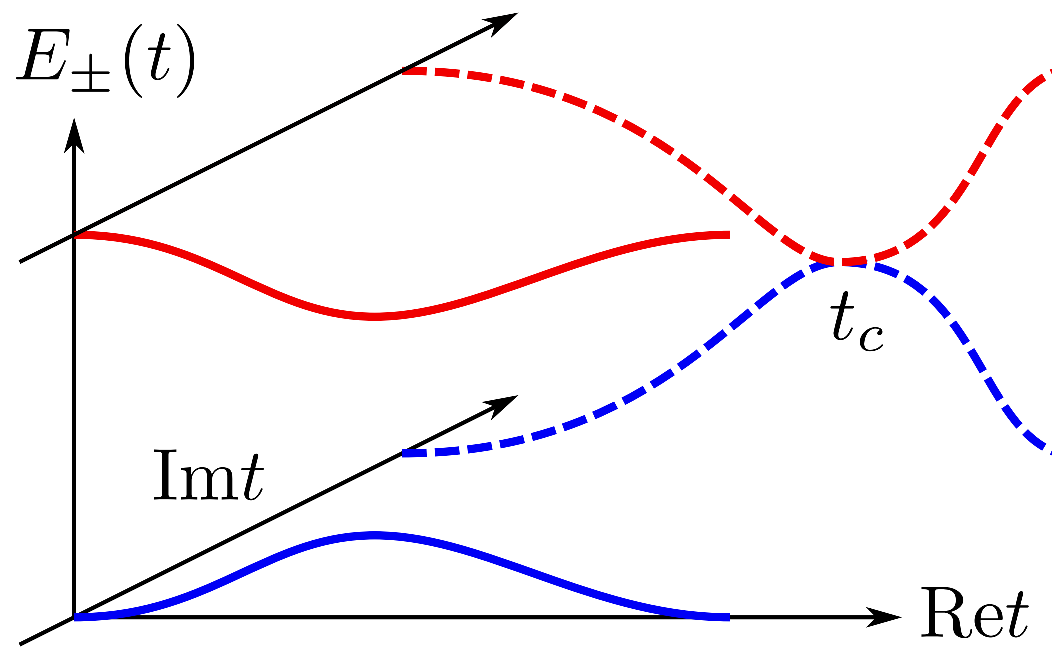
For $t \in \mathbb{R}\,, \delta E(t) \equiv E_+(t) - E_-(t) \neq 0\,.$

However, there can exist closing points $t_c \in \mathbb{C}$ such that $\delta E(t_c) = 0\,.$

For example, when $A(t) = -Et\,, \delta E(t) = 2\sqrt{\left(k_z + eEt\right)^2 + m_\perp^2}\,.$

$$\delta E(t_c) = 0 \Rightarrow t_c = -\frac{k_z}{eE} \pm i\frac{m_\perp}{eE}$$

The DDP formula makes use of t_c such that $\mathrm{Im}\, t_c > 0\,.$



DDP Approximation Formula

$$P_{\mathrm{DDP}} = \exp\left[-2\mathrm{Im}\int_0^{t_c} \delta E(t)dt\right] \equiv \exp\left[-2\mathrm{Im}\Delta(t_c)\right]$$

Although it is approximation scheme,
it surprisingly gives the analytical result for LZ model.

$$P(\boldsymbol{k}) = \exp\left(-\frac{\pi m_\perp^2}{eE}\right) = P_{\mathrm{DDP}}$$

$$\left[\begin{array}{l} \text{When } A(t) = -Et\,, \delta E(t) = 2\sqrt{\left(k_z + eEt\right)^2 + m_\perp^2} \text{ and } t_c = -\dfrac{k_z}{eE} + i\dfrac{m_\perp}{eE} \\ \text{Then, } -2\mathrm{Im}\Delta(t_c) = -4\mathrm{Im}\displaystyle\int_0^{t_c} dt\sqrt{\left(k_z + eEt\right)^2 + m_\perp^2} = -\dfrac{\pi m_\perp^2}{eE}. \end{array}\right]$$

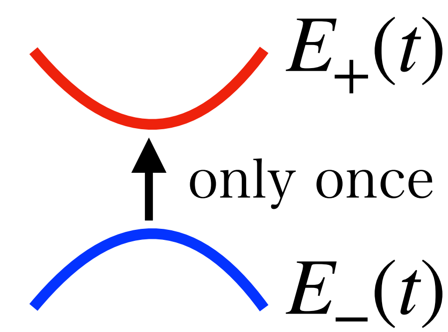
“Derivation” of DDP Formula

1. We expand the solution of $i\partial_t\psi = H\psi$ as $\psi = \sum_{i=\pm} a_i \psi_i e^{-iE_i t}$.

   The square of a transition amplitude $a_+(\infty)$ gives the probability , $P = \left| a_+(\infty) \right|^2$.

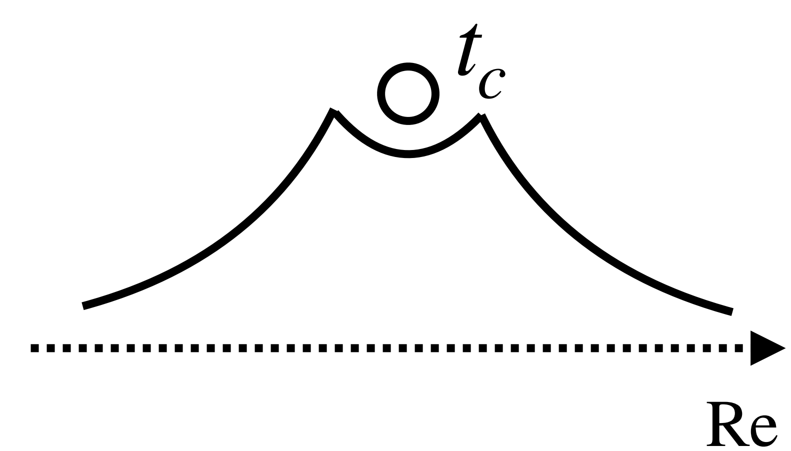
2. We derive coupled equation for $a_\pm(t)$ and solve them using first order truncation .

$$a_+(\infty) \simeq \int_{-\infty}^{\infty} dt \exp\left[\mathrm{i}\Delta(t) + \ln\eta(t)\right] \text{ where } \eta(t) = \psi_-^*(t)\dot{\psi}_+(t) \,.$$



3. We change the original integration contour in order to pick up contribution from t_c .

$$a_+(\infty) \simeq \exp[\mathrm{i}\Delta(t_c)] \left\{ \Rightarrow P = \exp[-2\mathrm{Im}\Delta(t_c)] \right\}$$



Contour transformation reminds us the Lefschetz-thimble method.

Lefschetz-Thimble Method

We employ the semiclassical approximation for $Z = \int dz \exp\left[-S(z)\right]$.

1. We find all saddle points, $z_{s,i}$. $\left.\frac{\partial S}{\partial z}\right|_{z=z_i} = 0$

thimbles

2. We draw (dual) thimbles defined by the flow equation.

$$\frac{dz}{d\tau} = \pm \overline{\frac{\partial S}{\partial z}} \quad \text{with} \quad z(\tau = 0) = z_{s,i}$$

Good properties of the flow

$$\frac{d}{d\tau}\mathrm{Im}\, S[z(\tau)] = 0\,, \frac{d}{d\tau}\mathrm{Re}\, S[z(\tau)] \geq 0\,(\,\leq 0)$$

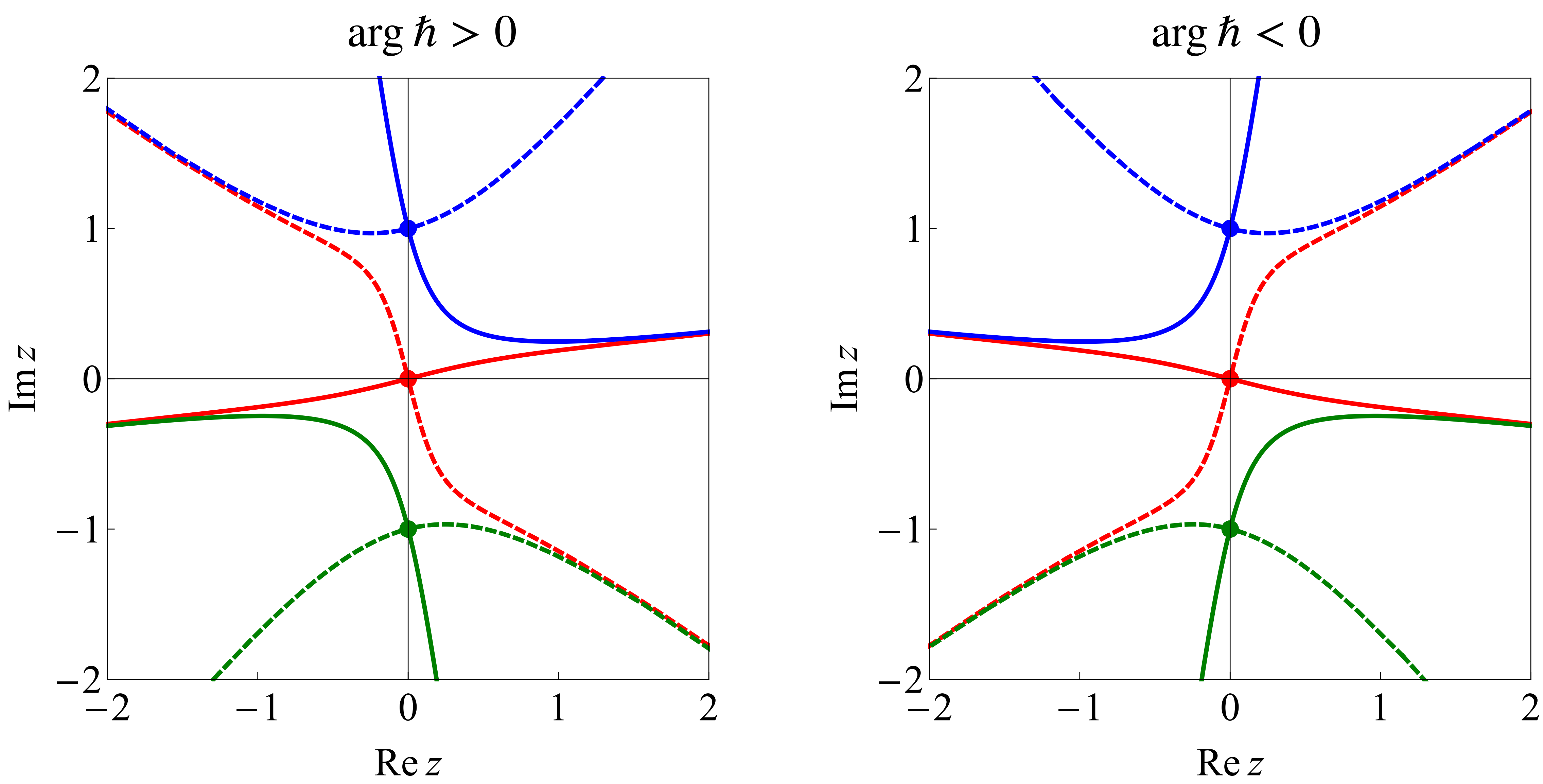
3. The thimbles contributing to Z are determined by the intersection #.

= Their dual thimbles intersect the original integration contour.

Examples of Lefschetz Thimbles

$$Z(\hbar) = \int dz \exp[-S(z)/\hbar] \text{ where } S(z) = z^2/2 + z^4/4 \,. \;\; \Rightarrow \;\; z_{s,i} = 0, \pm\, i$$

We complexify $\hbar$ and focus on the blue line.



The Stokes phenomenon occurs at arg $\hbar = 0$.

Lefschetz-Thimble Inspired Method

We apply the Lefschetz-thimble method to

$$a_+(\infty) \simeq \int_{-\infty}^{\infty} dt \exp\left[\mathrm{i}\Delta(t) + \ln \eta(t)\right] \equiv \int_{-\infty}^{\infty} dt \exp[-S(t)]\,.$$

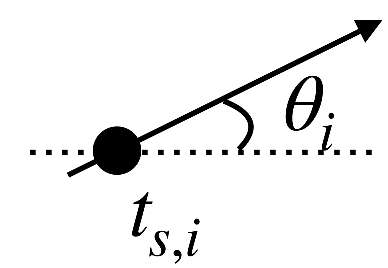
We use the Gaussian approximation to get

$$\boxed{a_+^{\mathrm{LT}}(\infty) = \sum_i n_i e^{i\theta_i - S(t_{s,i})} \sqrt{\frac{2\pi}{|S''(t_{s,i})|}}\,.}$$

$t_{s,i}$: i-th saddle point

n_i : i-th intersection #

θ_i : angle of i-th thimble



cf. w/o $\ln \eta(t)$, $a_+(\infty)$ reproduces DDP formula.

$$\left.\frac{d\Delta}{dt}\right|_{t=t_c} = \delta E(t_c) = 0 \Rightarrow a_+^{\mathrm{DDP}}(\infty) = \exp\left[i\Delta(t_c)\right]$$

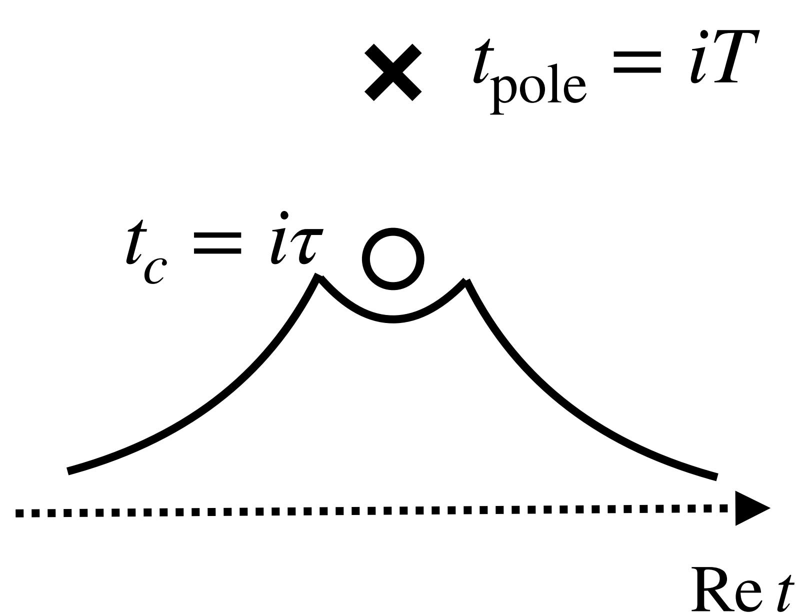
Modified Landau-Zener Model

We consider the following model.

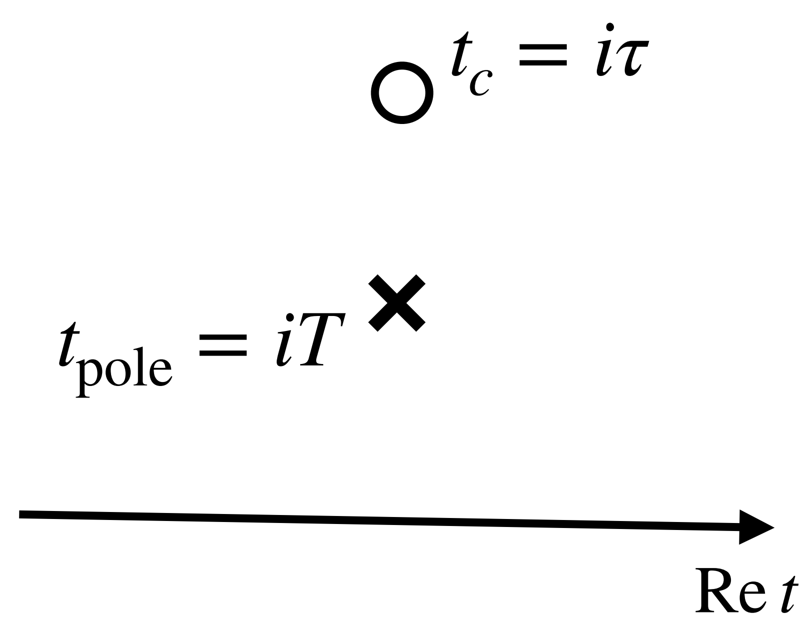
$$H(t) = \frac{\Lambda}{2\tau\sqrt{1+(t/T)^2}}\begin{pmatrix} t/\tau & 1 \\ 1 & -t/\tau \end{pmatrix}, \quad \delta E(t) = \frac{T\Lambda}{\tau^2}\sqrt{\frac{t^2+\tau^2}{t^2+T^2}}.$$

- It reduces to the Landau-Zener model in the limit of $T \to \infty$.
- There exist not only $t_c = i\tau$ but $t_{\text{pole}} = iT$.

When $\tau < T$, DDP works.

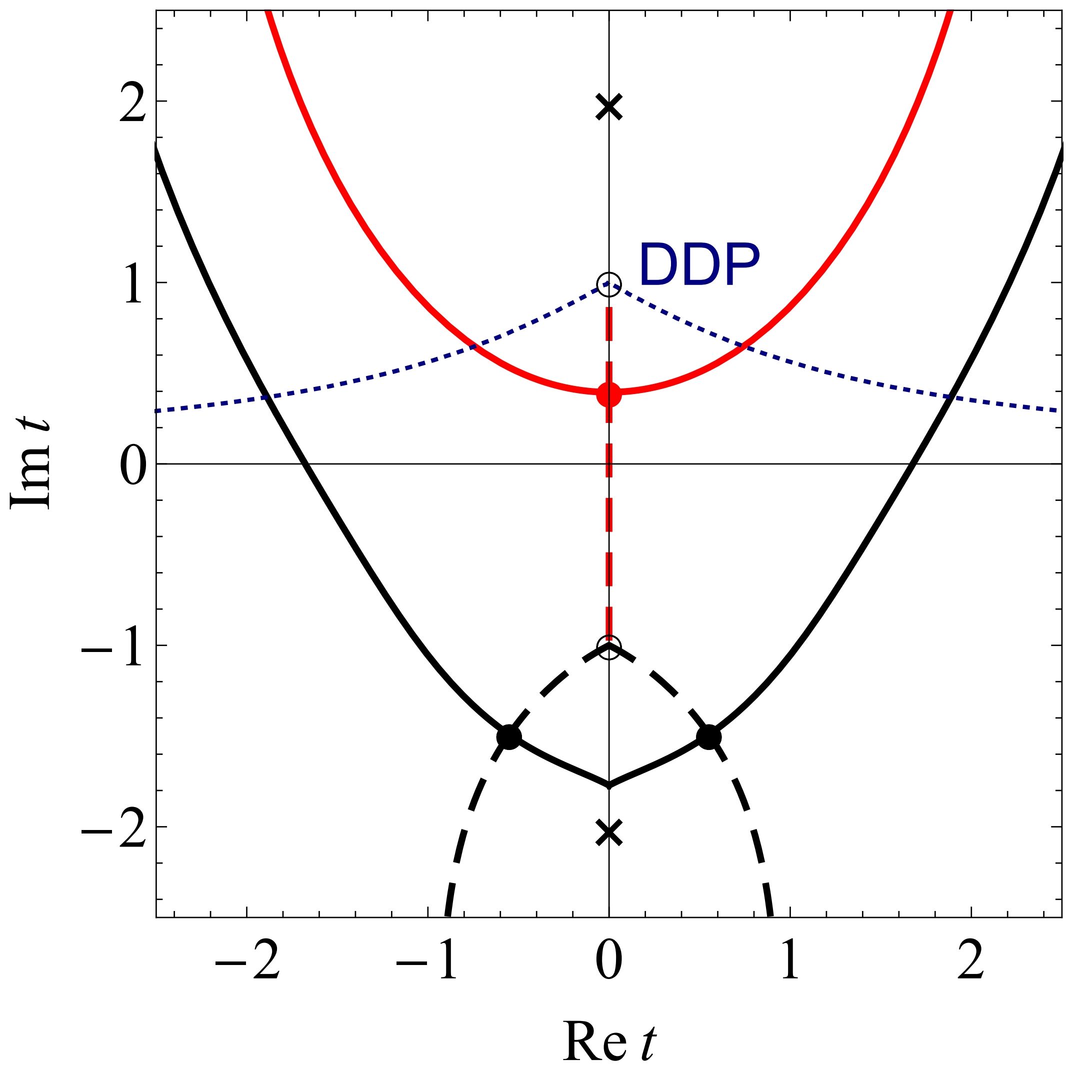


When $\tau \geq T$, DDP fails to work.

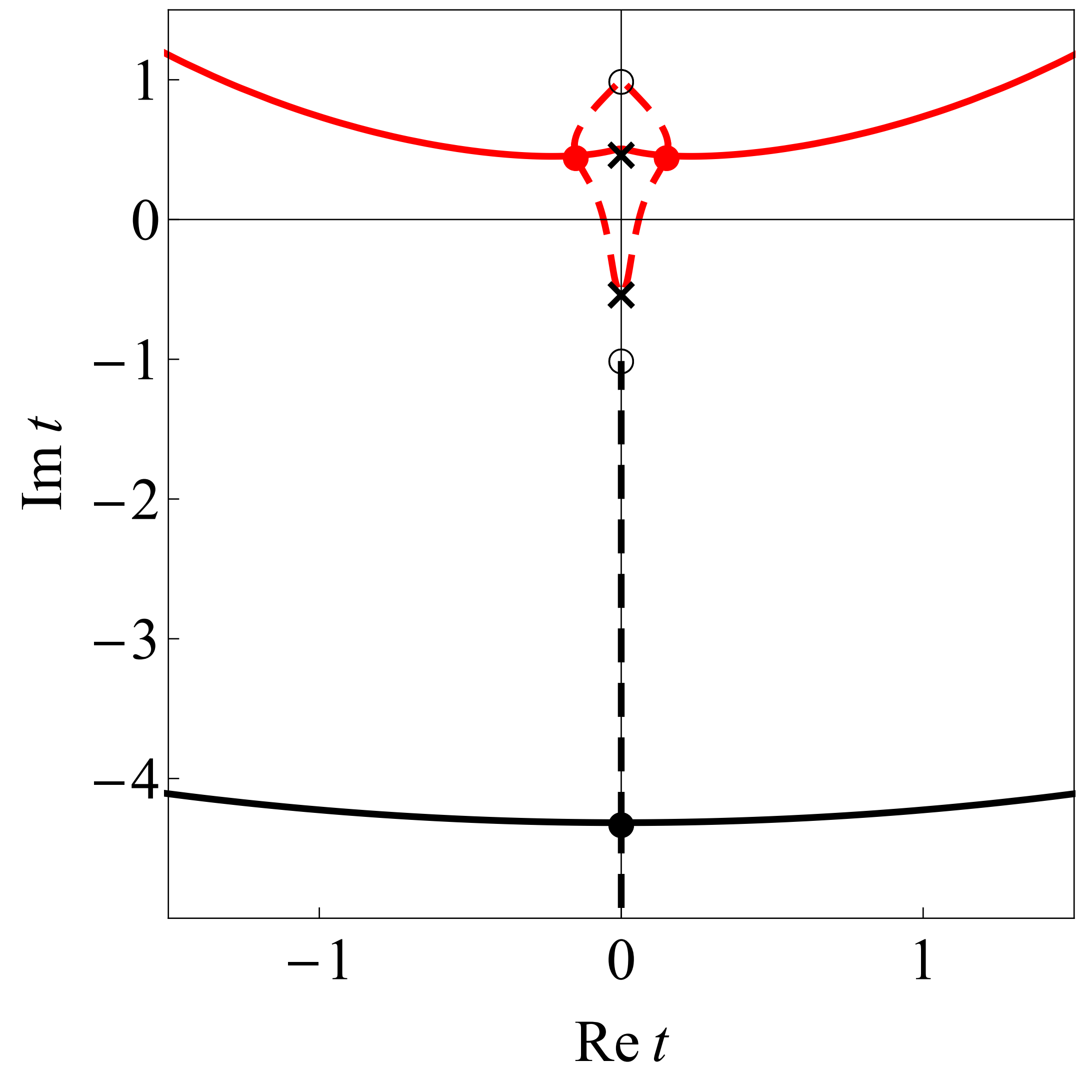


Thimble Structure

$\tau < T$



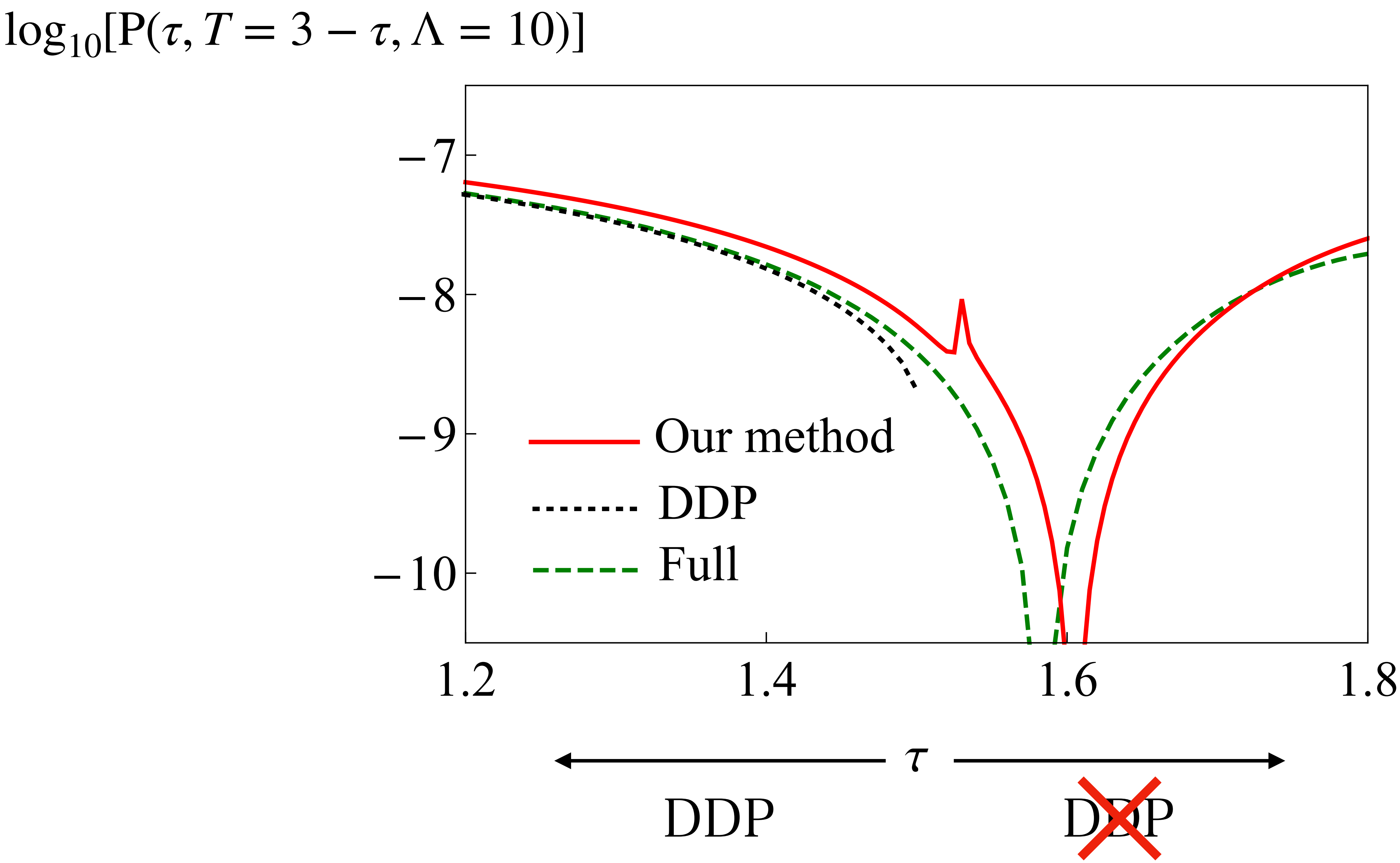
$\tau \geq T$



Dots $\begin{cases} \text{Cross: poles} \\ \text{Filled: saddle points} \\ \text{Open: closing points} \end{cases}$ Lines $\begin{cases} \text{Solid: thimbles} \\ \text{Dashed: dual thimbles} \\ \text{Dotted: DDP contour} \end{cases}$

The red thimble only contribute.

Comparison of Two Methods



- $\tau = 1.5$ is a boundary for DDP .
- Fake peak occurs due to Gaussian approximation.

Schwinger Mechanism Revisited

The Hamiltonian describing Schwinger mech.

$$H = \begin{pmatrix} k_z - eA(t) & m_\perp \\ m_\perp & -k_z + eA(t) \end{pmatrix}$$

When $A(t) = -Et$, it reduces the Landau-Zener model.

We consider a Sauter-type field and define the Keldysh parameter.

$$E_z = \frac{E}{\cosh^2 \omega t} \left[\Rightarrow A(t) = -\frac{E}{\omega} \tanh \omega t \right], \ \gamma \equiv \frac{m\omega}{eE}.$$

There exist a closing point $t_c = \frac{1}{\omega} \tanh^{-1}\left(-\frac{\gamma k_z}{m} + i \frac{\gamma m_\perp}{m} \right)$ and a pole $t_{\mathrm{pole}} = i \frac{\pi}{2\omega}$.

Sauter-Type Field

We define A by $P(\boldsymbol{k}) \equiv \exp\left(-Am^2/eE\right)$.

$$\gamma \equiv \frac{m\omega}{eE}$$

- For small γ, DDP works very well.

$$A_{\mathrm{DDP}} \simeq \frac{\pi m_\perp^2}{2m}\left(\frac{1}{m+\gamma k_z}+\frac{1}{m-\gamma k_z}\right)$$

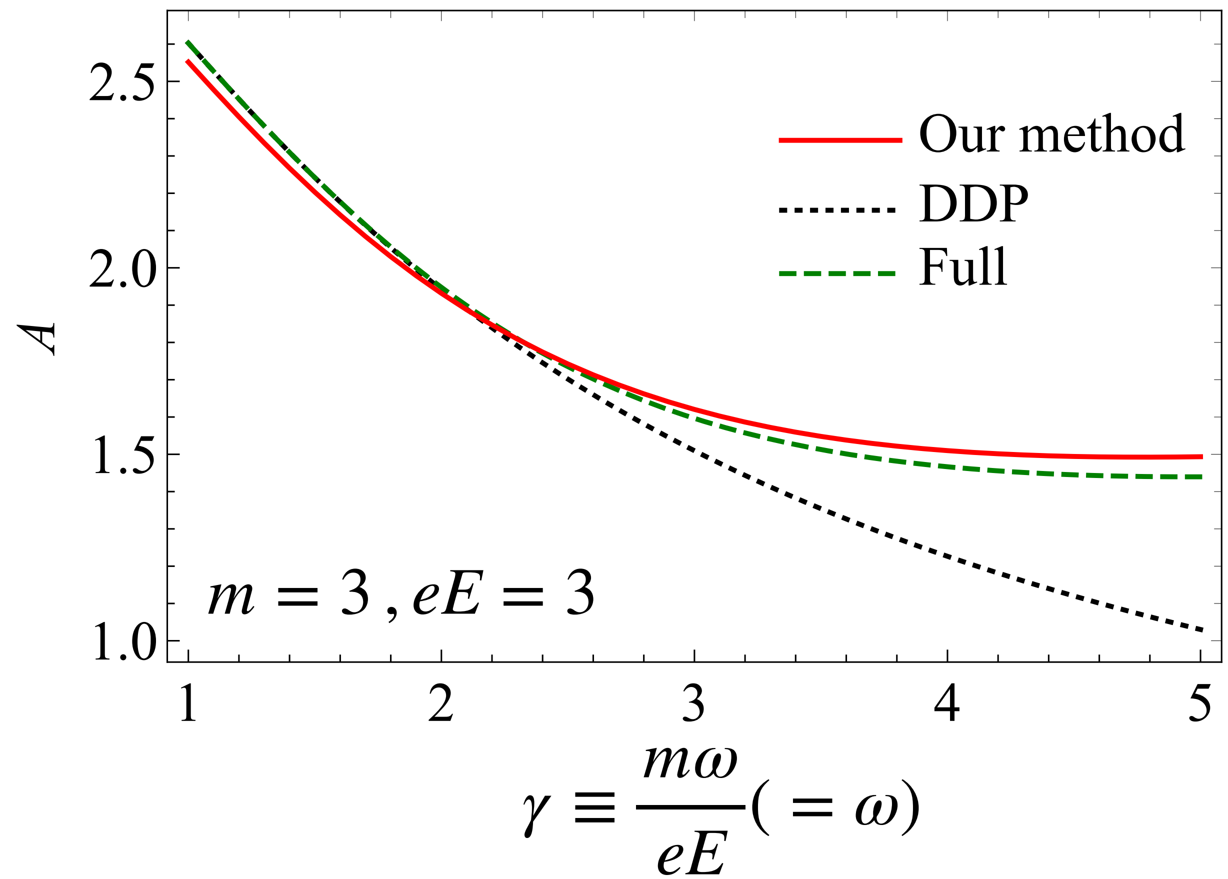
$$\begin{cases} A \to \dfrac{\pi m_\perp^2}{m^2} \text{ as } \gamma \to 0 \\ A \to \infty \text{ as } k_z \to \pm\dfrac{m}{\gamma} = \pm\dfrac{eE}{\omega} \end{cases}$$

- For large γ, DDP fails.

For simplicity, we set $\boldsymbol{k} = 0$.

Then, $t_c = i\dfrac{1}{\omega}\tan^{-1}\gamma$, $t_{\mathrm{pole}} = i\dfrac{\pi}{2\omega}$.

When $\gamma \to \infty$, t_c gets closer to t_{pole} .



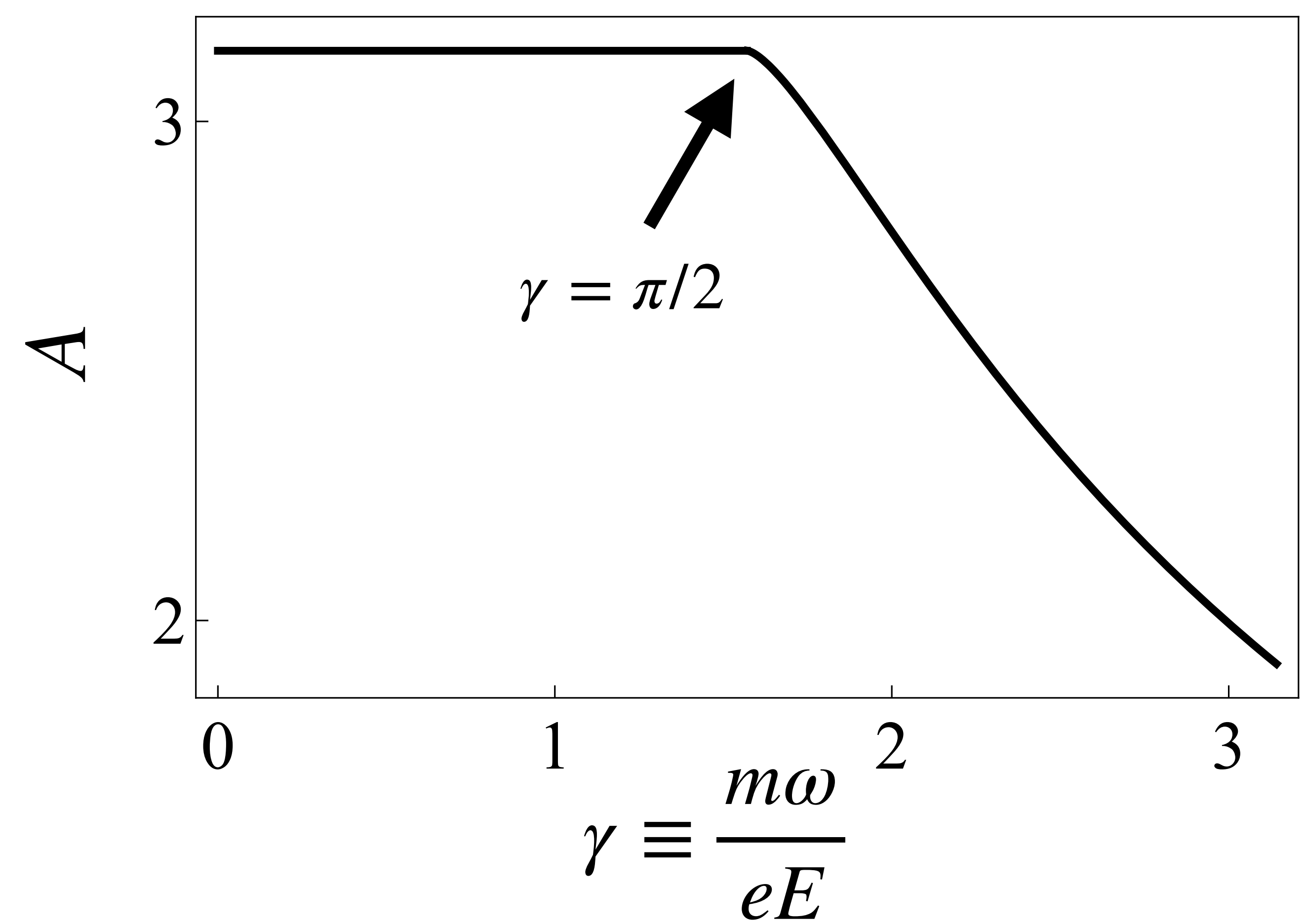
Dynamically Assisted Schwinger Mech.

We consider superposition of const. + Sauter-type field.

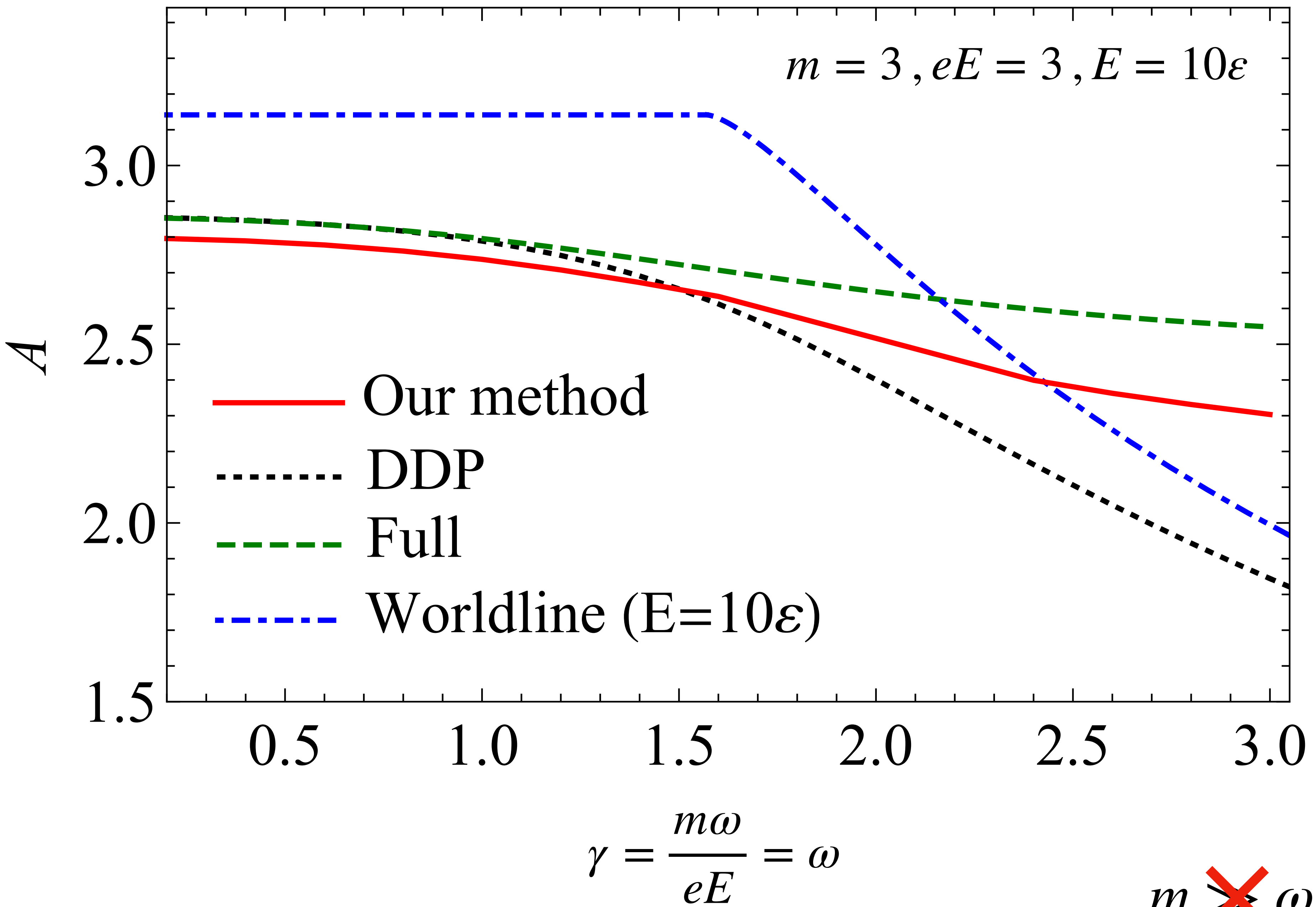
$$E_z = E + \frac{\varepsilon}{\cosh^2 \omega t} \quad \left[\Rightarrow A(t) = -Et - \frac{\varepsilon}{\omega} \tanh \omega t \right] \qquad (E \gg \varepsilon)$$

The worldline instanton gives the decay width as $\Gamma \sim \exp\left(-\frac{Am^2}{eE} \right)$.

- $m \gg \omega\,, m^2 \gg eE$ are assumed.
- Poles of $\tanh(\omega t)$ play pivotal roles.



Dynamically Assisted Schwinger Mech.



$m \cancel{\gg} \omega\,, m^2 \gg eE$

- DDP works well for small γ .
- DDP asymptotically approaches to Worldline.
- Our method always gives reasonable answer.

Summary & Prospect

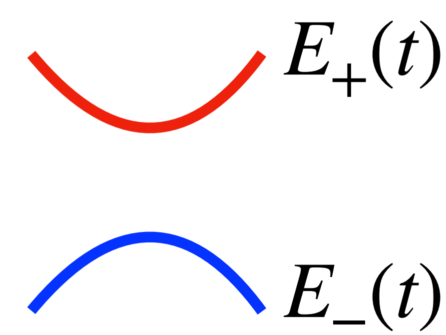
! Formulation of quantum tunneling

$$a_+^{\mathrm{LT}}(\infty) = \sum_i n_i e^{i\theta_i - S(t_{s,i})} \sqrt{\frac{2\pi}{|S''(t_{s,i})|}}$$

$$\mathrm{cf.}\ a_+^{\mathrm{DDP}}(\infty) = \exp\left[i\Delta(t_c)\right]$$

! Application to the Schwinger mechanism

- We dealt with it as two-level systems.
- We apply our method comparing with DDP.



? Stokes phenomenon in tunneling effects