TQFT, integrable lattice model, and quiver gauge theories

Toshihiro Ota
Osaka university, RIKEN iTHEMS

Part1 (math) : Based on the seminar series with mathematicians at iTHEMS
(Many thanks to Hokuto, Masaki, Kenta and Genki!)

Part2 (phys) : Based on the joint work with Kazunobu Maruyoshi and Junya Yagi,
"Wilson-'t Hooft lines as transfer matrices," [2009.12391]

Aim of Part1

I would like to introduce the followings to general audience in a mathematical language:

— What is quantum field theory (QFT)?

— What is lattice model?

— What does integrable mean?

In turn, the 2nd part of my talk is on
"an integrable lattice model emerges from a certain class of quantum gauge theories."

(I also hope these will lead to a good exchange among us,
especially between physicists and mathematicians.)

Contents

- What is quantum field theory (QFT)?
- Lattice model as discrete QFT
- Integrable lattice model

1-page introduction to quantum mechanics

Let's start from quantum mechanics (QM)
(it actually turns out to be 1d QFT)

Suppose we have a 1d mfd (time) M^1 = interval or S^1 (or $\mathbb{R}$), and data to specify a QM:

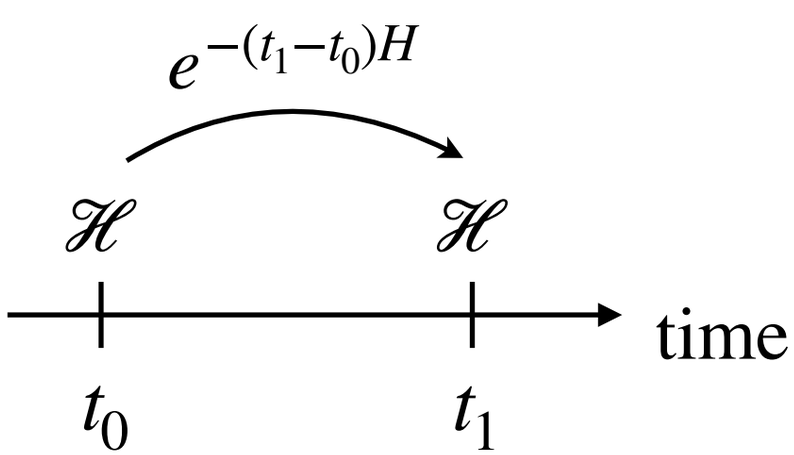
— $\mathscr{H}$: a vector space (state space)

— $\mathscr{O}$: a set of self-adjoint operators (observables, acting on $\mathscr{H}$)

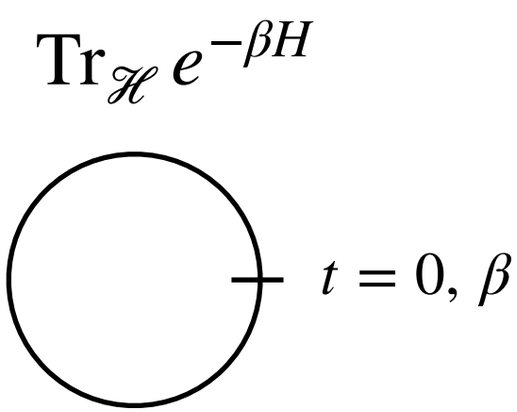
— $H\,(\in \mathscr{O})$: an operator called *Hamiltonian*

Two most crucial properties of QM (Comment : I do not get into "imaginary time")

1. For an interval $M^1 = [t_0, t_1]$,
we have a linear map (time evolution)



2. For a circle $M^1 = S^1_\beta$,
we get a number (partition function)



1-page introduction to quantum mechanics

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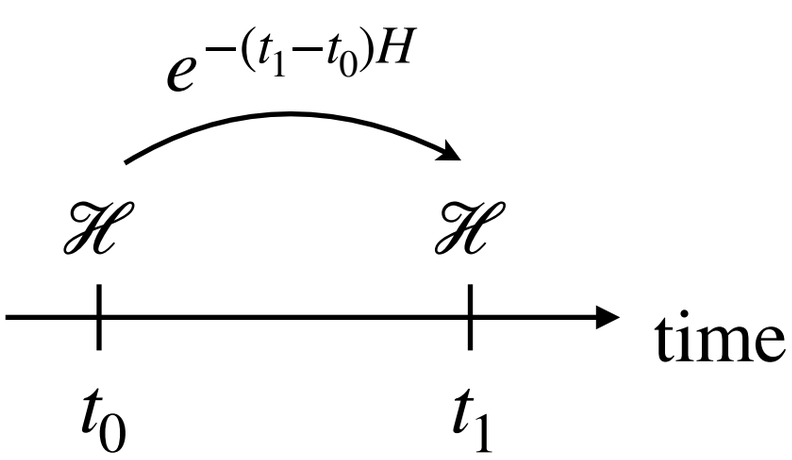
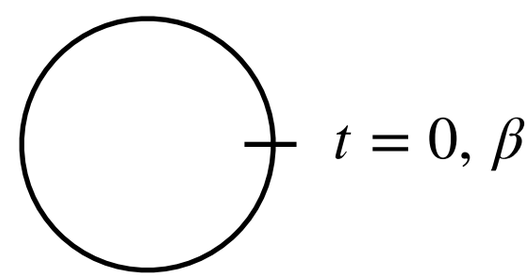


2. For circle $M^1 = S^1_\beta$,
we get a number (partition function)

$$\mathrm{Tr}_{\mathscr{H}}\, e^{-\beta H}$$

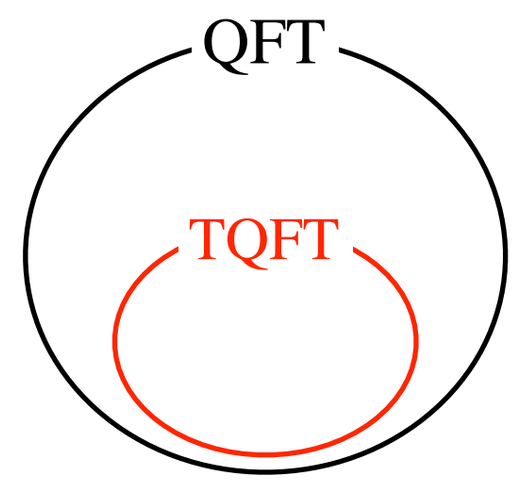


We expect a QFT in general dimensions M^{d+1} also has the properties:

— Given a manifold with boundary,
produces vectors at the boundary & linear maps among them

— Given a closed manifold, produces a number

— Compatible with cutting & gluing the manifolds



What is QFT?

Atiyah axiomatized *topological* QFT from the physical inputs ('88).

<u>Axiom</u> ((d+1)-dimensional TQFT)

A (d+1)-dim. TQFT is defined by Z_T satisfying the following properties (1), (2), and (A1)-(A5):

(1) For a closed d-dim. mfd N^d, associate a <u>finite</u> dim. $\boldsymbol{C}$-vector space $\mathscr{H}_N$ (state space):

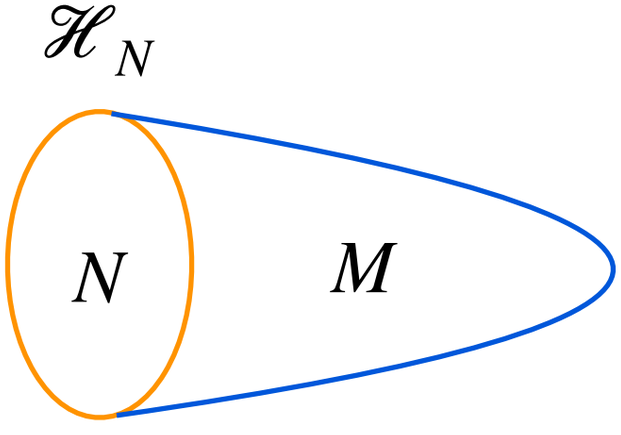
$$Z_T(N^d) = \mathscr{H}_N$$

1/2 of Canonical quantization

(2) For a (d+1)-dim. mfd M^{d+1}, $\partial M^{d+1} = N^d$, associate a vector (state):

$$Z_T(M^{d+1}) \in \mathscr{H}_N$$

Path integral quantization



Atiyah's TQFT axiom

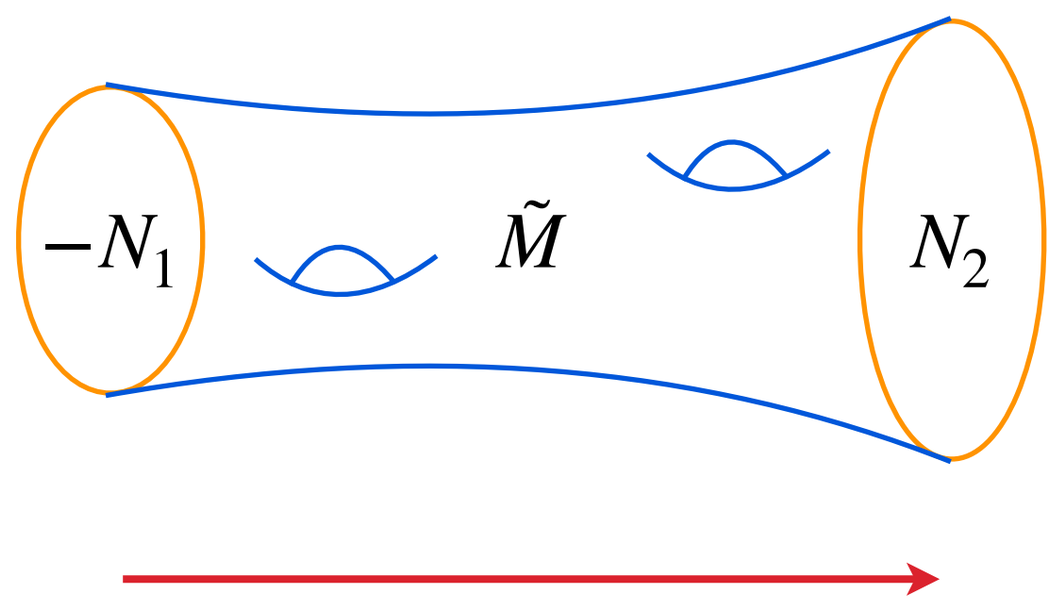
(A1) (involutory) $\mathscr{H}_{-N} = \mathscr{H}_N^*$

(A2) (multiplicative) $\mathscr{H}_{N_1 \sqcup N_2} = \mathscr{H}_{N_1} \otimes \mathscr{H}_{N_2}$

From the conditions so far, for a (d+1)-dim. mfd $\tilde{M}$, $\partial\tilde{M} = -N_1 \sqcup N_2$, we have a linear map:

$$Z_T(\tilde{M}) \in \mathrm{Hom}_{\mathbb{C}}(\mathscr{H}_{N_1}, \mathscr{H}_{N_2})$$

i.e. a cobordism between d-dim. mfds defines the time evolution



Atiyah's TQFT axiom

(A3) $Z_T(\emptyset) = \mathbb{C}$. (—> for $\partial M = \emptyset$, $Z_T(M) \in \mathbb{C}$. Partition function)

(A4) For cobordisms M_1, M_2, s.t. $\partial M_1 = -N_1 \sqcup N_2$ and $\partial M_2 = -N_2 \sqcup N_3$,

$$Z_T(M_1 \cup_{N_2} M_2) = Z_T(M_2)\, Z_T(M_1)$$

(A5) For an interval I, the linear map is <u>trivial</u>:

$$Z_T(I \times N) = \mathrm{id}_{\mathscr{H}_N}$$

In topological theory, the time evolution is trivial for an interval.
(*e.g.* (2+1)-d Chern-Simons theory is a TQFT and its Hamiltonian $\equiv 0$.)

Atiyah's TQFT axiom

All in all, Atiyah's (d+1)-dim. TQFT Z_T^{d+1} defines a functor from

$$\begin{aligned} \text{Obj} &= \text{d-dim. closed manifolds} \\ \text{Mor} &= \text{cobordisms} \end{aligned}$$

to

$$\begin{aligned} \text{Obj} &= \text{finite dim. } \boldsymbol{C}\text{-vector spaces} \\ \text{Mor} &= \text{linear maps} \end{aligned}$$

$$Z_T^{d+1} : \text{Bord}_{d+1} \longrightarrow \text{Vect}_{\mathbb{C}}$$

For more general QFTs (*i.e.* not TQFT), the axiom needs to be somehow modified.
In general, a *model* is (expected to be) defined by some functor Z.

Contents

- ☑ What is quantum field theory (QFT)?

  A functor Z, which gives a number for a closed mfd : $Z(M) \in \mathbb{C}$.
  (Specify Z = Define a model)

- ☐ Lattice model as discrete QFT

- ☐ Integrable lattice model

What is lattice model?

(d+1)-dim. QFT :
for a (d+1)-dim. closed mfd M, defines a number (partition function)

$$Z_{\mathrm{Q}}^{d+1}(M) \in \mathbb{C}$$

(d+1)-dim. lattice model :
for a (d+1)-dim. lattice L,
defines a number (again this is called partition function)

$$Z_{\mathrm{L}}^{d+1}(L) \in \mathbb{C}$$

"Lattice model is discrete QFT"

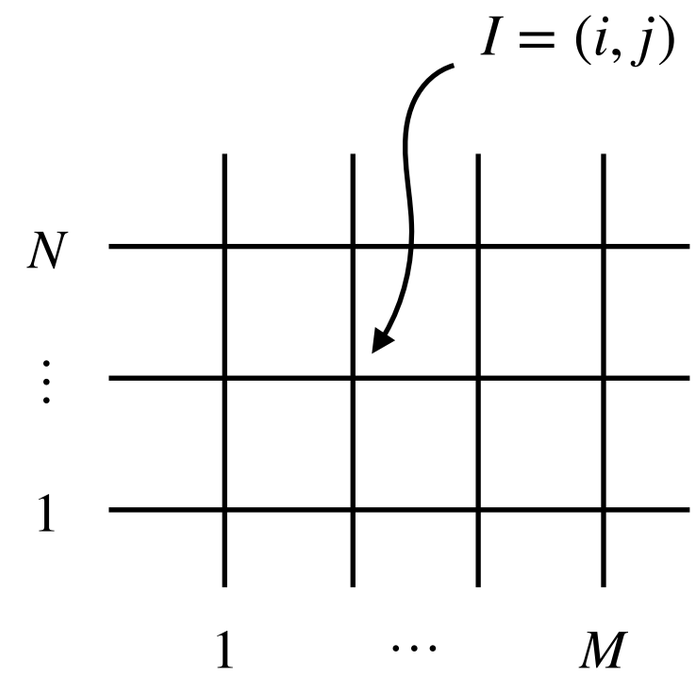
In this talk I focus on $d = 0$ or 1 case only.

Prominent example : the Ising model

Setup : 2d lattice & spin configuration

<u>2d lattice on torus $\mathbb{T}^2$</u>

$$L := \mathbb{Z}_M \times \mathbb{Z}_N$$
$$= \{1, \cdots, M\} \times \{1, \cdots, N\}$$
$$I = (i \bmod M,\ j \bmod N) \in L$$



<u>Spin configuration on the lattice L</u>

is a map :

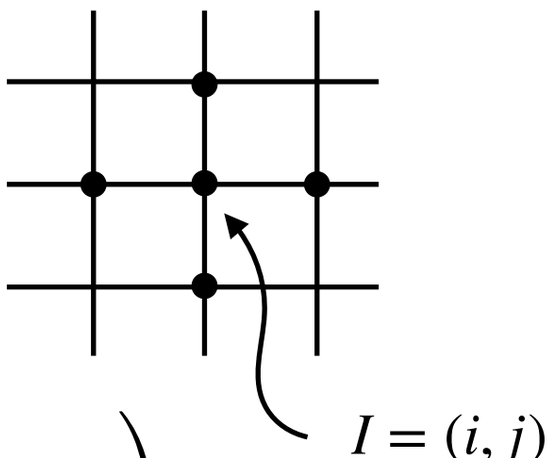
$$\mathbf{s} : L \to \{+1, -1\}$$

+1 : up spin, -1 : down spin

Let $S(L)$ be the space of all the spin configurations.

Prominent example : the Ising model

Define <u>the energy functional of 2d Ising model</u> by



$$E_{\text{Ising}} : S(L) \to \mathbb{R}$$

s.t.

$$E_{\text{Ising}}(\mathbf{s}) = -\sum_{\langle I, I' \rangle} s_I s_{I'} := -\left(\sum_{i,j=1}^{M,N} s_{ij} s_{i,j+1} + \sum_{i,j} s_{ij} s_{i+1,j} \right)$$

$$s_{ij} = s_I := \mathbf{s}(I)$$

One of the simplest models of spin system; only the nearest neighbor spins have interaction.
(Comment for physicists : here I set J=1)

<u>The partition function of 2d Ising model</u> is defined by

Planck's constant^{-1}

$$Z(L, E_{\text{Ising}}; \beta) := \sum_{\mathbf{s} \in S(L)} e^{-\beta E_{\text{Ising}}(\mathbf{s})}$$

closed mfd

path integral

Action/Lagrangian

Prominent example : the Ising model

$$Z(L, E_{\mathrm{Ising}}; \beta) := \sum_{\mathbf{s} \in S(L)} e^{-\beta E_{\mathrm{Ising}}(\mathbf{s})}$$

$\beta \in \mathbb{R}_{\geq 0}$, and the summand is generically called *Boltzmann weight*.

To discuss integrability, we need to introduce another quantity called the *free energy* in the thermodynamic limit $M, N \to \infty$:

$$-\beta F(\beta) := \lim_{N,M \to \infty} \frac{1}{NM} \log Z(\beta)$$

<u>Physical FACT</u>

For a general statistical model, any thermodynamical quantities (in principle) can be obtained from the free energy.

Contents

- ☑ What is quantum field theory (QFT)?

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- ☑ Lattice model as discrete QFT

  Lattice model is defined by the "discrete version" of partition function Z

- ☐ Integrable lattice model

Integrable lattice model

Finally I should say something more about

— What does “integrable” mean?

— How about operator/Hamiltonian formalism in lattice model?

Unfortunately, there is no canonical definition of “integrability” in this context. From the physical fact, we may say a model is integrable (solvable) if one can compute the exact free energy:

$$-\beta F(\beta) := \lim_{N,M\to\infty} \frac{1}{NM} \log Z(\beta)$$

As it turns out, Integrable
= Can compute an exact free energy
= Can compute an exact partition function
(= Can diagonalize the transfer matrix)

To explain these, let us consider 1d Ising model and introduce *transfer matrix*

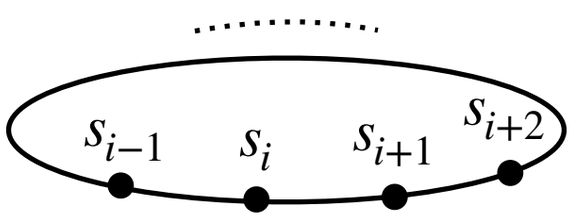
Integrable lattice model

Define <u>the energy functional of 1d Ising model</u> by

$$E_{\text{1d Ising}}(\mathbf{s}) := -\sum_{i=1}^{N} s_i s_{i+1}$$

$$\mathbf{s} : L_{1d} \to \{\pm 1\}$$

$$L_{1d} = \mathbb{Z}_N = \{1, \cdots, N\}$$



<u>The partition function of 1d Ising model</u> is given by

$$Z(L_{1d}, E_{1d}; \beta) := \sum_{\mathbf{s} \in S(L_{1d})} e^{-\beta E_{1d}(\mathbf{s})}$$

$$= \sum_{s_1, \ldots, s_N = \pm 1} e^{\beta s_1 s_N} \cdots e^{\beta s_3 s_2} e^{\beta s_2 s_1}$$

Integrable lattice model

Introduce a 2x2 matrix, called *transfer matrix* :

$$T = \left(e^{\beta s s'}\right)_{s,s'=\pm 1} = \begin{pmatrix} e^{\beta} & e^{-\beta} \\ e^{-\beta} & e^{\beta} \end{pmatrix}$$

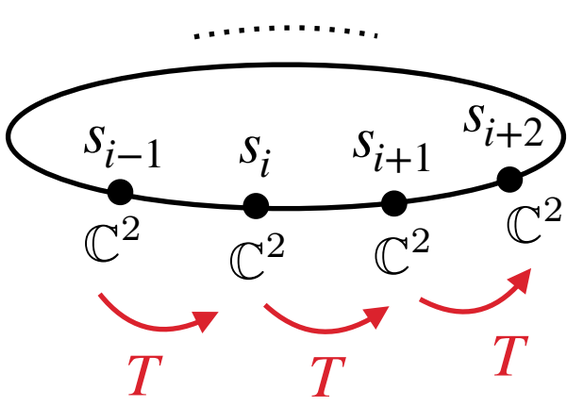
Then the partition function becomes

$$\begin{aligned} Z(\beta) &= \sum_{s_1,\ldots,s_N=\pm 1} e^{\beta s_1 s_N} \cdots e^{\beta s_3 s_2} e^{\beta s_2 s_1} \\ &= \sum_{s_1,\ldots,s_N=\pm 1} T_{s_1 s_N} \cdots T_{s_3 s_2} T_{s_2 s_1} \\ &= \sum_{s_1=\pm 1} \left(T^N\right)_{s_1 s_1} = \mathrm{Tr}_{\mathbb{C}^2}\, T^N \end{aligned}$$

quantum state space
of 1d Ising spin chain

We have the state space $\mathbb{C}^2$ at the boundary of each 1d segment, and the transfer matrix sends a state to the adjacent site, which is the "time evolution" of this system:

$$\log T \propto \text{Hamiltonian}$$



Integrable lattice model

$$Z(\beta) = \mathrm{Tr}_{\mathbb{C}^2} T^N \,, \quad T = \begin{pmatrix} e^{\beta} & e^{-\beta} \\ e^{-\beta} & e^{\beta} \end{pmatrix}$$

From this expression, we immediately obtain the exact free energy from the largest eigenvalue of T : let the eigenvalues be $\lambda_1 > \lambda_2$,

$$\begin{aligned} -\beta F(\beta) &= \lim_{N\to\infty} \frac{1}{N} \log Z(\beta) \\ &= \lim_{N\to\infty} \frac{1}{N} \log \lambda_1^N \left(1 + \left(\frac{\lambda_2}{\lambda_1} \right)^N \right) \\ &= \log \lambda_1 = \log(2\cosh\beta) \end{aligned}$$

Integrable
= Can compute an exact free energy
= Can compute an exact partition function
= Can diagonalize the transfer matrix

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  Integrable = Can compute an exact free energy (partition function)

To be continued to the 2nd part

Plan of the 2nd part

- Integrability revisited, in the language of physics
- Elliptic & trigonometric transfer matrix from L-operators
- Wilson-’t Hooft lines as transfer matrices
- Emergence of integrability from TQFT in extra dimensions

TQFT, integrable lattice model, and quiver gauge theories

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Part2 (phys) : Based on the joint work with Kazunobu Maruyoshi and Junya Yagi, "Wilson-'t Hooft lines as transfer matrices," [2009.12391]

Aim of Part2

To explain the correspondence : [Maruyoshi-TO-Yagi]

Gauge theory side *Lattice model side*

Wilson-'t Hooft lines = transfer matrices

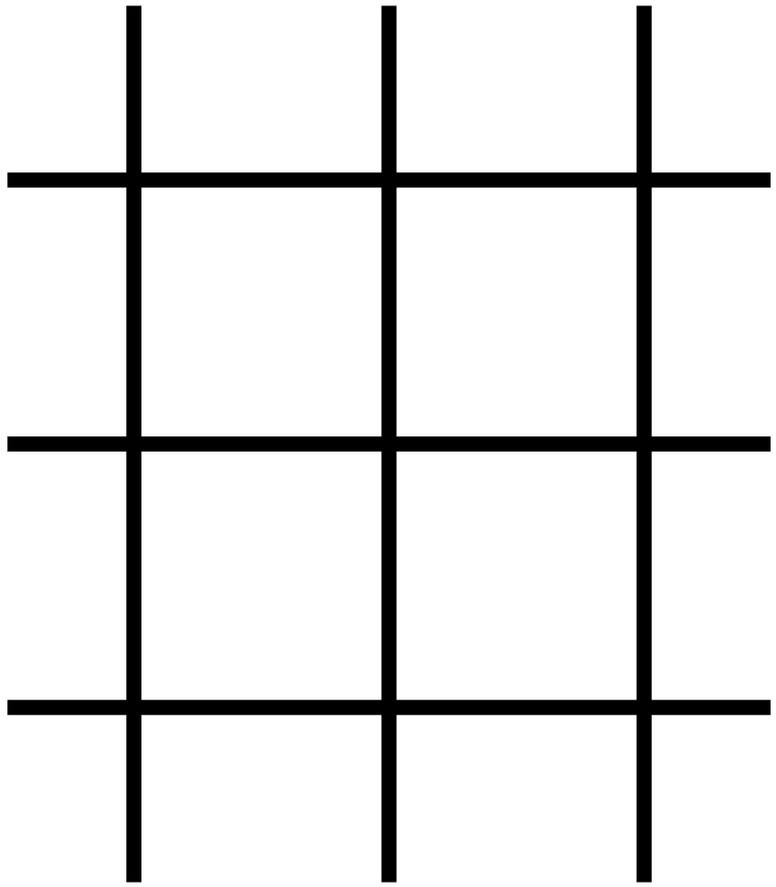
- $\mathcal{N} = 2$ circular quiver theory on $S^1 \times \mathbb{R}^3$
- Wrap a Wilson-'t Hooft line T around S^1
- $\langle T \rangle$ is a function of Coulomb branch parameters
- Quantization of $\langle T \rangle$ coincides with the transfer matrix of trigonometric quantum integrable system
- Related to other correspondences via string dualities

Notice

The details might be too technical, you can ignore them.
Today's message is the above correspondence, that's all.

Introduction

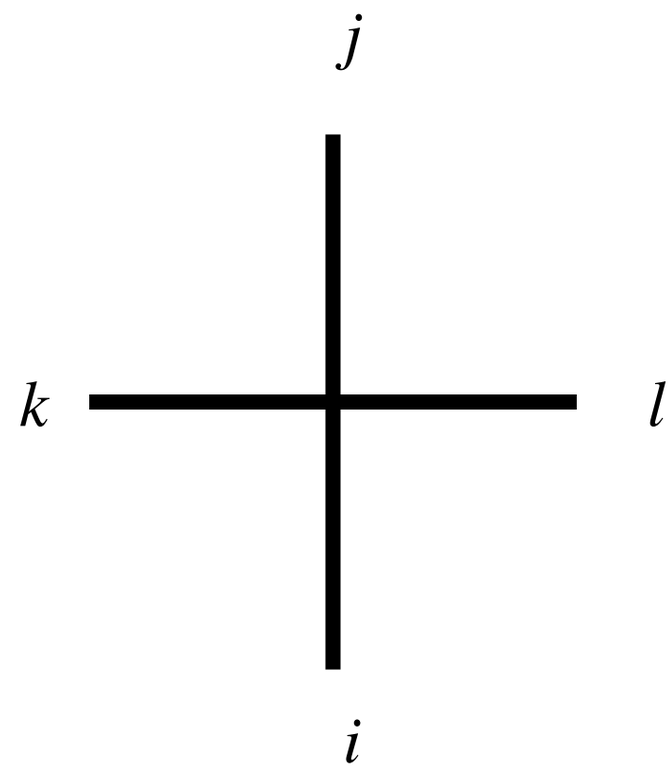
2d lattice model



Introduction

2d lattice model

$$B^{ij}_{kl}$$

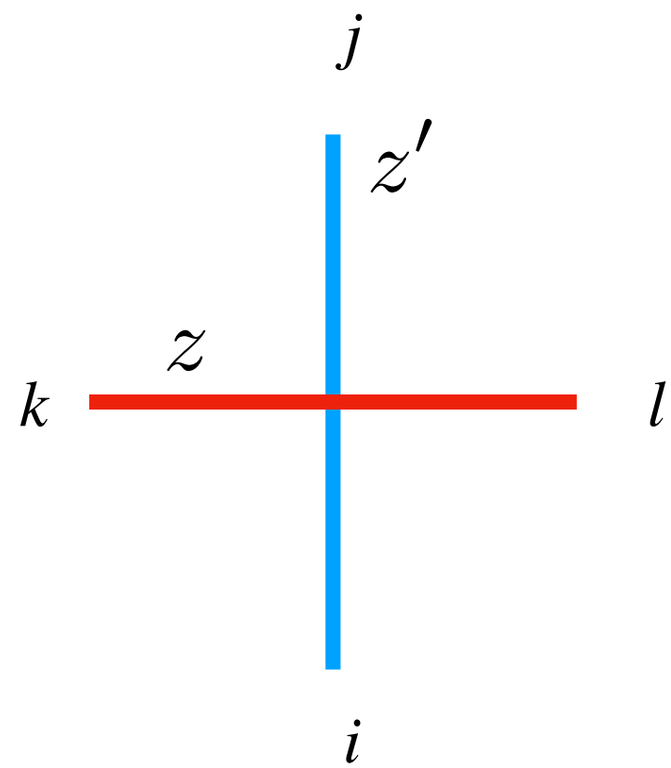


Boltzmann weight

Introduction

2d lattice model

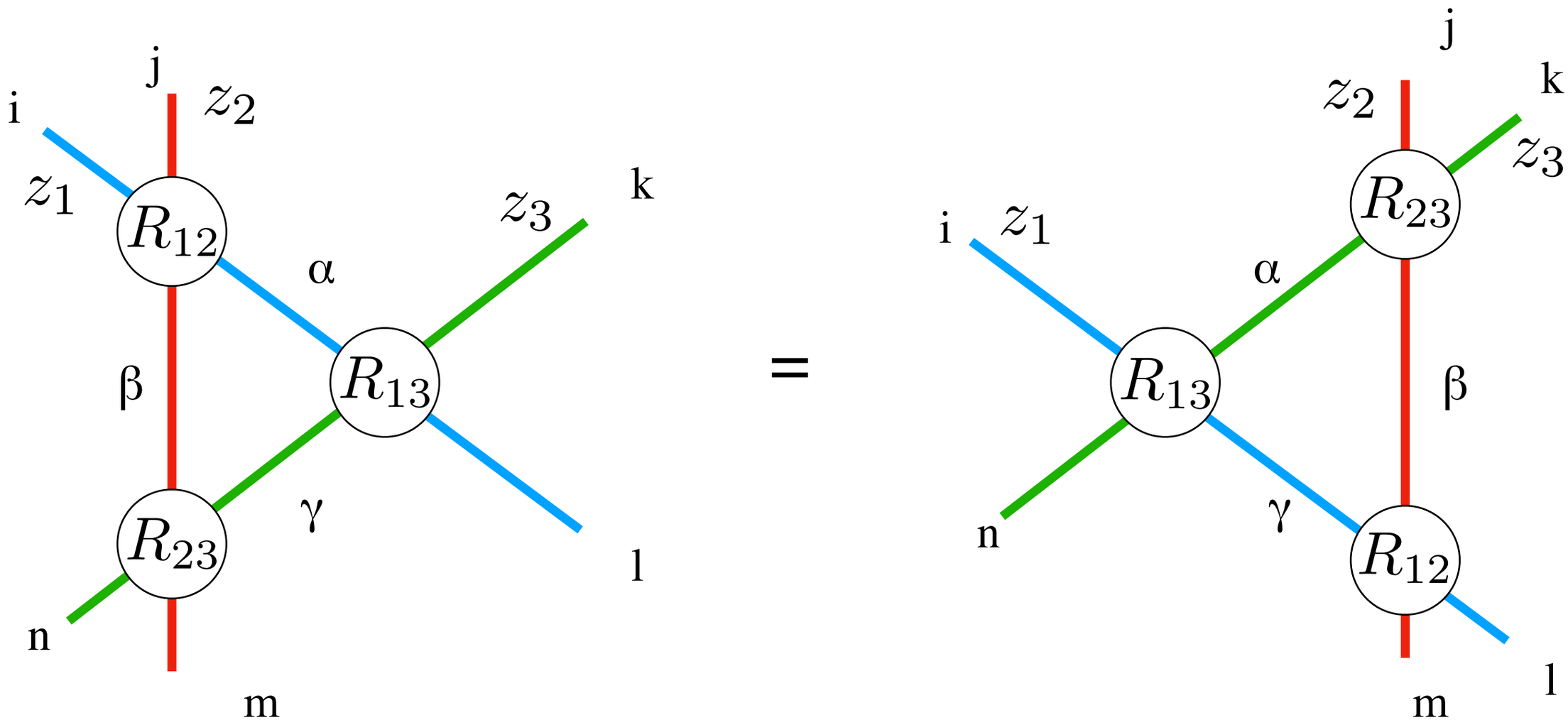
$$R^{ij}_{kl}(z, z')$$



R-matrix = Boltzmann weight
w/ spectral parameter z

Introduction

Integrability = Yang-Baxter equation



$$\sum_{\alpha,\beta,\gamma} R^{i\alpha}_{j\beta}(z_1,z_2)R^{\alpha l}_{k\gamma}(z_1,z_3)R^{\beta m}_{\gamma n}(z_2,z_3) = \sum_{\alpha,\beta,\gamma} R^{j\beta}_{k\gamma}(z_2,z_3)R^{i\alpha}_{\gamma n}(z_1,z_3)R^{\alpha l}_{\beta m}(z_1,z_2)$$

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- Integrability revisited, in the language of physics

- Elliptic & trigonometric transfer matrix from L-operators

  Lattice model side

- Wilson-’t Hooft lines as transfer matrices

  Gauge theory side

- Emergence of integrability from TQFT in extra dimensions

Today’s message

“Wilson-’t Hooft lines = transfer matrices”

The details might be too technical, you can ignore them.

Classical integrability

= Can exactly solve an EoM

We know classical integrable systems such as

Harmonic oscillator

$$H = \frac{1}{2}p^2 + \frac{1}{2}x^2$$

Kepler problem

$$H = \frac{1}{2}(p_1^2 + p_2^2) - \frac{k}{r}$$

But why?

Classical integrability

= Can exactly solve an EoM

> Theorem (Liouville)
> In n dimensional system, if there exist n independent conserved quantities, then the eom can be exactly solved.

1d Harmonic oscillator $$H = \frac{1}{2}p^2 + \frac{1}{2}x^2$$

—> energy E

2d Kepler problem $$H = \frac{1}{2}(p_1^2 + p_2^2) - \frac{k}{r}$$

—> energy E, angular momentum L

How about field theory?

Classical integrability

= Can exactly solve an EoM

Integrable field theories: KdV, sine-Gordon, 2d Toda, …

E.g. KdV eq.
$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial^3 u}{\partial x^3} - 6u\frac{\partial u}{\partial x}$$

has infinite number of conserved quantities:

$$I_1 = \int u dx,$$
$$I_2 = \int u^2 dx,$$
$$I_3 = \int \left(2u^3 + (\partial_x u)^2\right) dx,$$
$$I_4 = \int \left(5u^4 + 10u(\partial_x u)^2 + (\partial_x^2 u)^2\right) dx,$$
$$\dots$$

Enough number of conserved quantities
= Large enough symmetry of the system

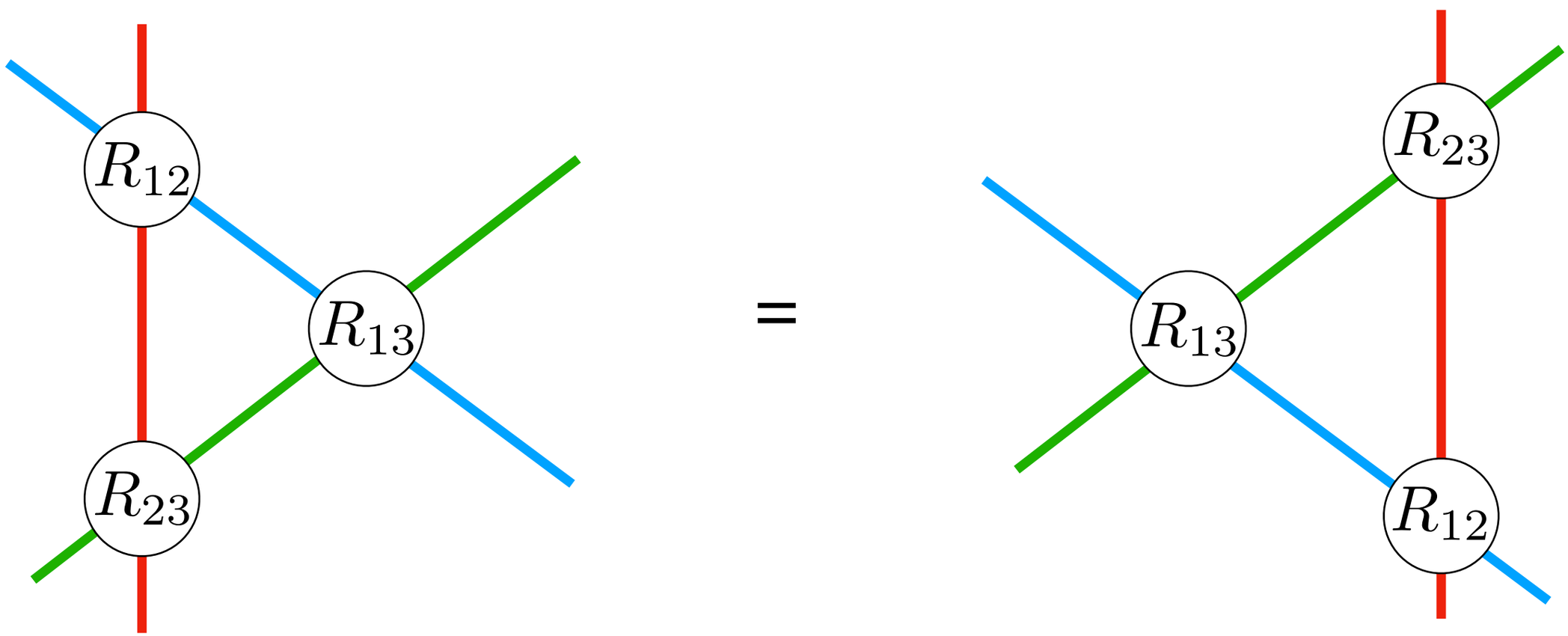
So far classical,
quantum integrability?

Quantum integrability

= Can compute an exact partition function
(or can diagonalize a Hamiltonian)

When?

R-matrix satisfying Yang-Baxter eq.



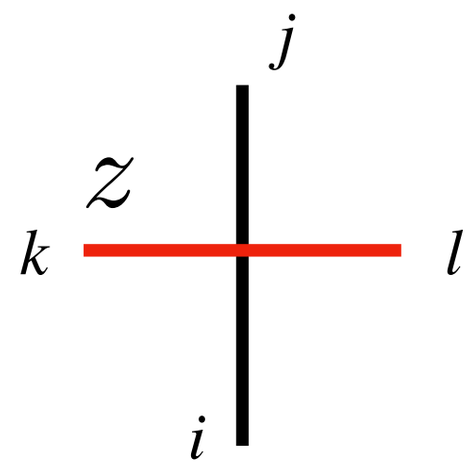
implies large enough symmetry of the system

2d lattice model

= discretized (Euclidean) QFT

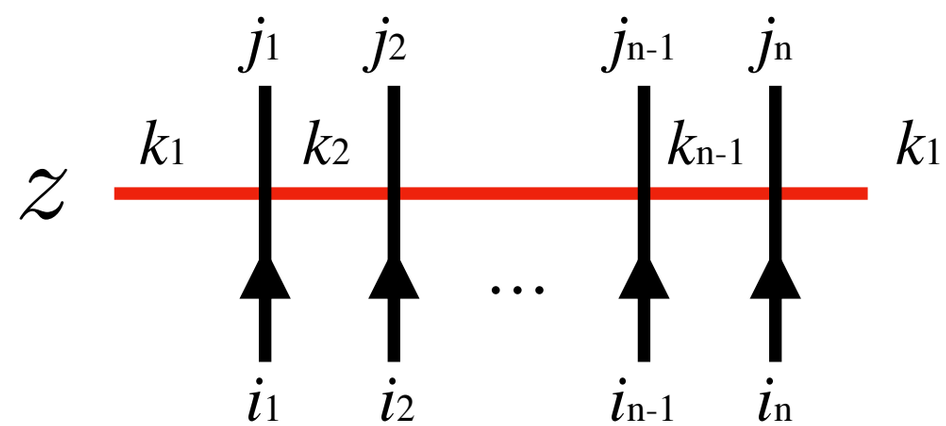
R-matrix (=Boltzmann weight)

$$R^{ij}_{kl}(z)$$



Transfer matrix (=time evolution from states *i*s to states *j*s)

$$T^{j_1\cdots j_n}_{i_1\cdots i_n}(z) = \sum_{\{k_i\}} R^{i_1 j_1}_{k_1 k_2}(z) R^{i_2 j_2}_{k_2 k_3}(z) \cdots R^{i_n j_n}_{k_{n-1} k_1}(z)$$



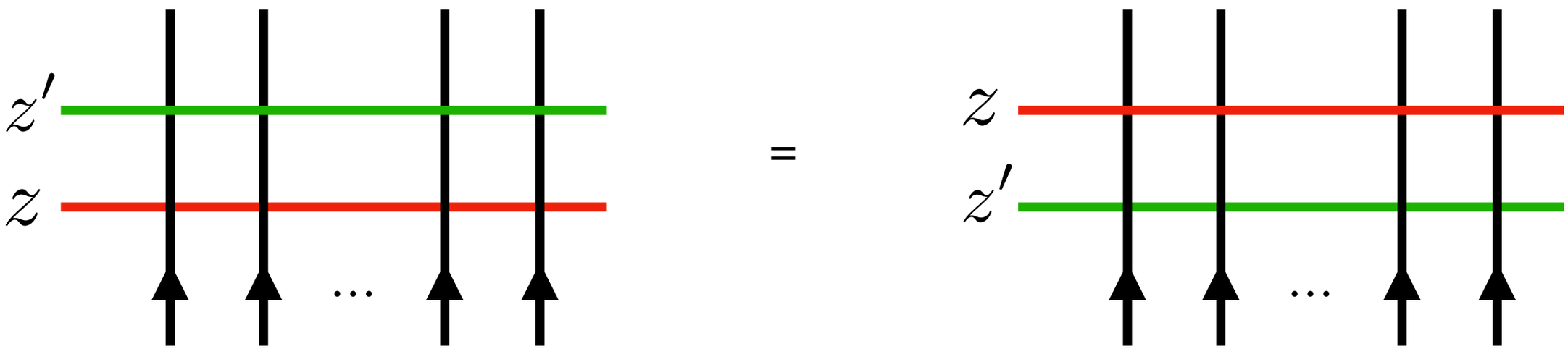
Partition function on nxm periodic lattice

$$\sum_{\{\text{config.}\}} \prod_{\{\text{vertices}\}} R^{ij}_{kl}(z) = \mathrm{Tr}\, T(z)^m$$

2d lattice model

= discretized (Euclidean) QFT

<u>FACT</u> From Yang-Baxter equation for $R^{ij}_{kl}(z)$ (+ unitarity condition), the transfer matrices with different spectral parameters commute:



<u>FACT</u> (Bethe ansatz)

$$[T(z), T(z')] = 0 \;\Rightarrow\; \text{Can obtain all the eigenvalues of } T(z)$$

Integrability
= Yang-Baxter eq. for $R^{ij}_{kl}(z)$
= commuting transfer matrix

Contents

- ☑ Integrability revisited, in the language of physics

  Quantum integrability = Commuting transfer matrices

- ☐ Elliptic & trigonometric transfer matrix from L-operators

  Lattice model side

- ☐ Wilson-'t Hooft lines as transfer matrices

  Gauge theory side

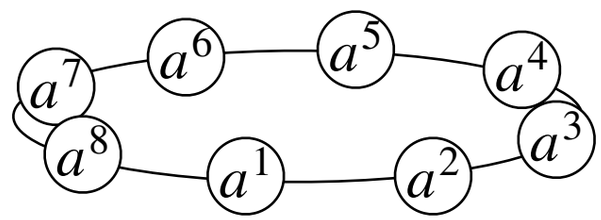
- ☐ Emergence of integrability from TQFT in extra dimensions

Today's message
"Wilson-'t Hooft lines = transfer matrices"
The details might be too technical, you can ignore them.

Transfer matrix from L-operators

Lattice model side

Consider a periodic quantum spin chain



Spins $a^1, \ldots, a^n \in \mathfrak{h}^*$, $\mathfrak{h} =$ Cartan of $\mathfrak{sl}_N$:

$$a^r = \mathrm{diag}(a_1^r, \ldots, a_N^r), \qquad \sum_{i=1}^{N} a_i^r = 0$$

a^r are called *dynamical parameters* in the integrable model literature.

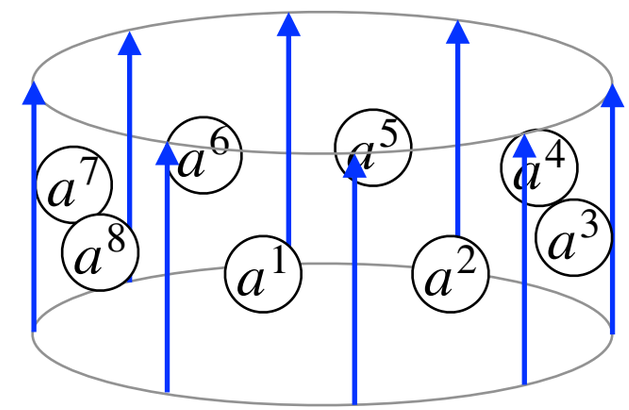
Local Hilbert space at each site : $\mathscr{M}_{\mathfrak{h}^*} = \{$meromorphic functions on $\mathfrak{h}^*\}$,

Total Hilbert space : $\mathscr{H} = \mathscr{M}_{\mathfrak{h}^*}^{\otimes n}$

Transfer matrix from L-operators

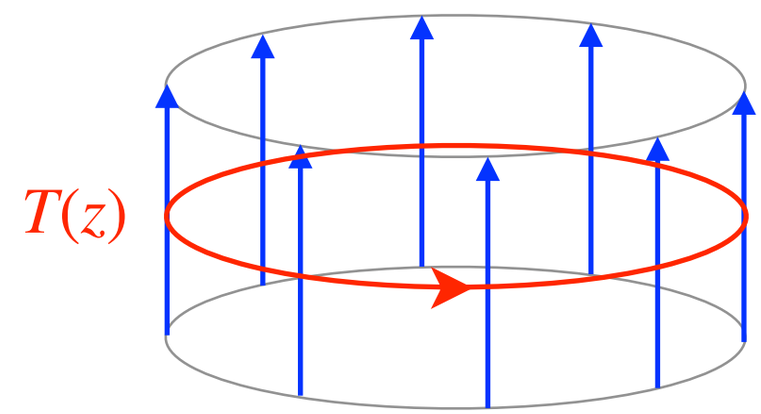
Lattice model side

Place the spin chains on a cylindrical lattice.



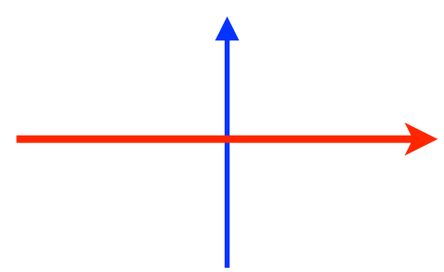
Transfer matrix is a horizontal loop operator,
thought of as the world-line of a particle whose
Hilbert space is $\mathbb{C}^N$; its state changes as it crosses other lines.

Suppose the particle depends on a continuous parameter,
which turns out to be a spectral parameter $z \in \mathbb{C}$.



Transfer matrix of two kinds of lines consists of
n copies of *L-operator* (a cousin of R-matrix) :

$L(z) \;=\; z$



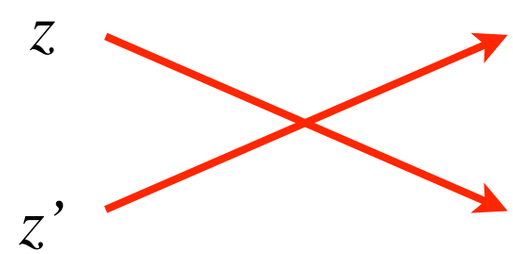
Transfer matrix from L-operators

Lattice model side

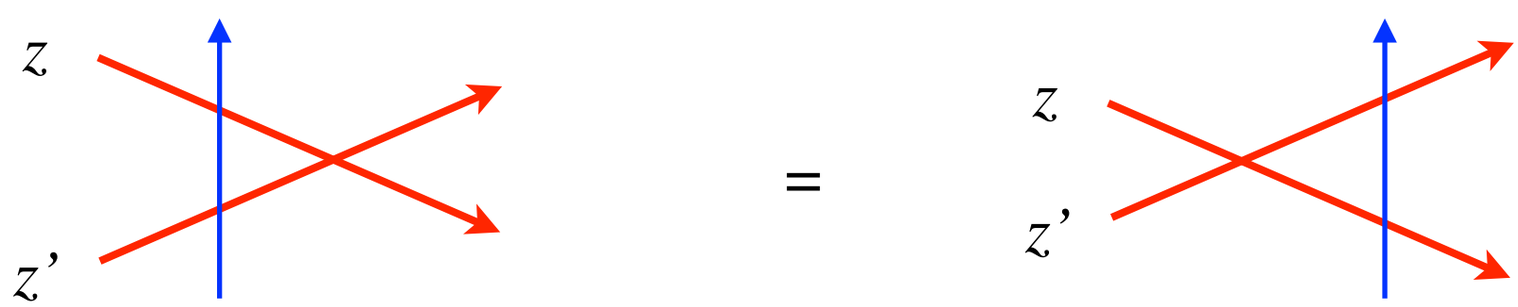
Two crossing red lines give R-matrix (scattering of the particles; S-matrix!) :

$$R : \mathbb{C} \times \mathbb{C} \longrightarrow \mathrm{End}(\mathbb{C}^N \otimes \mathbb{C}^N)$$

$R(z, z') \quad =$



Here, Yang-Baxter eq. is called *RLL relation*, from which it follows that the transfer matrices commute and the model becomes integrable:



<u>FACT</u> The elliptic L-operator and the dynamical R-matrix satisfy RLL relation.

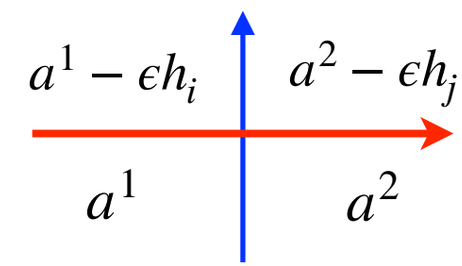
[Etingof-Varchenko, Ruijsenaars, Hasegawa]

Transfer matrix from L-operators

Lattice model side

The L-operator acts on $\mathscr{M}_{\mathfrak{h}^*} \otimes \mathscr{M}_{\mathfrak{h}^*}$ as difference operator and spins jump across the red lines:

$L(z) \;=\; z$

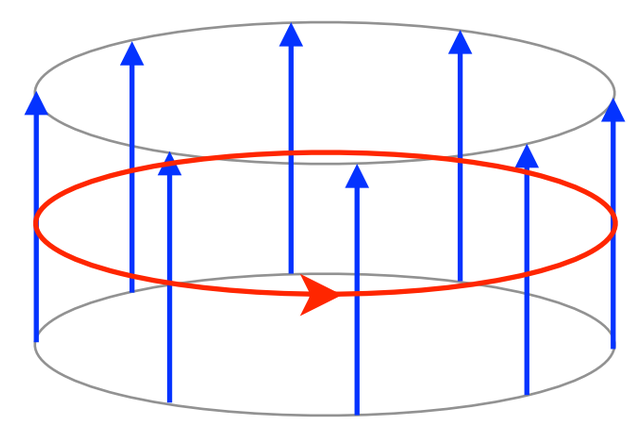


$$L(z) = \sum_{i,j} L\left(z; a^1, a^2\right)_i^j \Delta_i^1 \Delta_j^2, \quad \Delta_i^r : a^r \mapsto a^r - \epsilon h_i$$

$\epsilon \in \mathbb{C}$ is a constant parameter, $h_1 = (1 - 1/N, -1/N, \ldots, -1/N)$, etc.

The transfer matrix from n copies of L-operator acts on the total Hilbert space $\mathscr{H} = \mathscr{M}_{\mathfrak{h}^*}^{\otimes n}$, and defines the discrete time evolution of the spin chain:

$$T(z) = \sum_{i^1,\ldots,i^n} \prod_{r=1}^{n} L\left(z; a^r, a^{r+1}\right)_{i^r}^{i^{r+1}} \prod_{s=1}^{n} \Delta_{i^s}^s, \quad i^{n+1} = i^1$$



Transfer matrix from L-operators

Lattice model side

What remain to do, in order to compare to gauge theory side, are

1. Take trigonometric limit
2. Rewrite the transfer matrix by “more fundamental” L-operator

1. By trigonometric limit, the elliptic L-operator reduces to

$$L(z) \to \mathcal{L}_{w,m}(z) = \sum_{i,j} \left(\Delta_i^1 \Delta_j^2\right)^{\frac{1}{2}} \frac{\sin \pi \left(z - w + a_j^2 - a_i^1\right)}{\sin \pi (z - w)} \ell_m \left(a^1, a^2\right)_i^j \left(\Delta_i^1 \Delta_j^2\right)^{\frac{1}{2}}$$

$$\ell_m \left(a^1, a^2\right)_i^j = \left(\frac{\prod_{k(\neq i)} \sin \pi \left(a_k^1 - a_j^2 - m\right) \prod_{l(\neq j)} \sin \pi \left(a_i^1 - a_l^2 - m\right)}{\prod_{k(\neq i)} \sin \pi \left(a_{ki}^1 - \frac{1}{2}\epsilon\right) \sin \pi \left(a_{ik}^1 - \frac{1}{2}\epsilon\right)} \right)^{\frac{1}{2}}$$

This satisfies RLL relation with trigonometric dynamical R-matrix.
$w, m \in \mathbb{C}$ are spectral parameters assigned to blue lines.

Transfer matrix from L-operators

Lattice model side

2. Further, introduce fundamental L-operators

$$\mathcal{L}_{\pm,m} = \lim_{w \to \pm \mathrm{i}\infty} \mathcal{L}_{w,m}$$

then the trigonometric L-operator is written by

$$\mathcal{L}_{w,m}(z) = \frac{e^{\pi \mathrm{i}(z-w)} \mathcal{L}_{+,m} - e^{-\pi \mathrm{i}(z-w)} \mathcal{L}_{-,m}}{\sin \pi (z-w)}.$$

For each $\sigma \in \{\pm\}^n$ and $m \in \mathbb{C}^n$, construct the transfer matrix from the fundamental L-operators:

$$\mathcal{T}_{\sigma,m} = \sum_{i^1,\ldots,i^n} \left(\prod_{s=1}^{n} \Delta_{i_s}^s \right)^{\frac{1}{2}} \prod_{r=1}^{n} e^{\pi \mathrm{i} \sigma^r \left(a_{i^{r+1}}^{r+1} - a_{i^r}^r \right)} \ell_{m^r} \left(a^r, a^{r+1} \right)_{i^r}^{i^{r+1}} \left(\prod_{s=1}^{n} \Delta_{i_s}^s \right)^{\frac{1}{2}}$$

$$\Delta_i^r : a^r \mapsto a^r - \epsilon h_i$$

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- ☐ Wilson-’t Hooft lines as transfer matrices

  Gauge theory side

- ☐ Emergence of integrability from TQFT in extra dimensions

Today’s message

“Wilson-’t Hooft lines = transfer matrices”

The details might be too technical, you can ignore them.

Wilson-’t Hooft lines as transfer matrices

Gauge theory side

In general, 4d $\mathcal{N} = 2$ gauge theories have half-BPS Wilson-’t Hooft lines, which are the world-line of very massive dyonic particle with charge [Kapustin]

$$(\mathbf{m}, \mathbf{e}) \in \left(\Lambda_{\text{coweight}}(\mathfrak{g}) \times \Lambda_{\text{weight}}(\mathfrak{g})\right) \big/ \text{Weyl}$$

Wilson line has $\mathbf{m} = 0$ and is labeled by the representation of $\mathfrak{g}$,

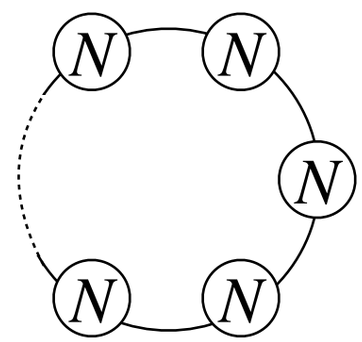
’t Hooft line has $\mathbf{e} = 0$ and is labeled by the representation of ${}^L\mathfrak{g}$.

Wilson-’t Hooft
= ’t Hooft (monopole background) + Wilson for subgroup of G leaving $\mathbf{m}$ invariant

Wilson-'t Hooft lines as transfer matrices

Gauge theory side

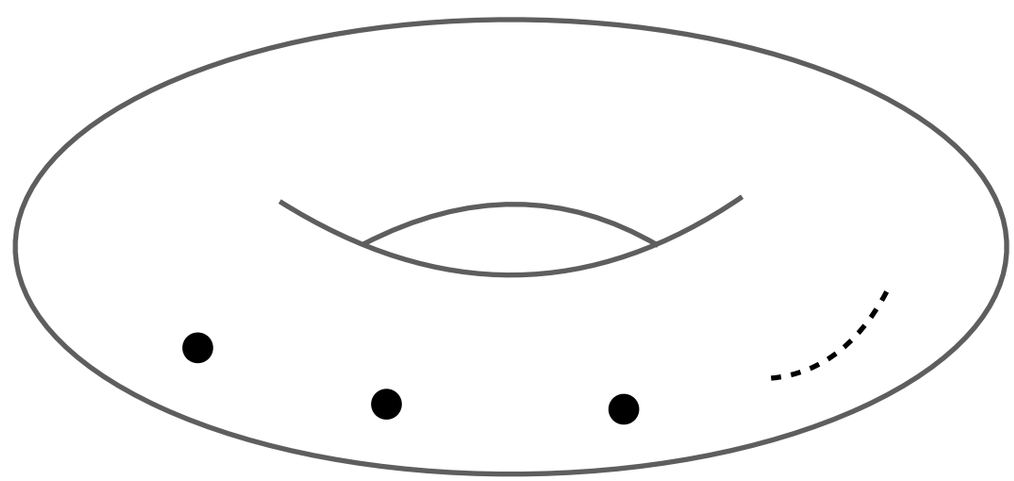
4d $\mathcal{N} = 2$ n-node circular quiver theory



Each node: SU(N) vector multiplet, edges: bifundamental hypers with masses $m^1, \ldots, m^n$

Such a theory is constructed from compactification of 6d (2,0) theory on n-punctured torus

[Gaiotto, Gaiotto-Moore-Neitzke, …]



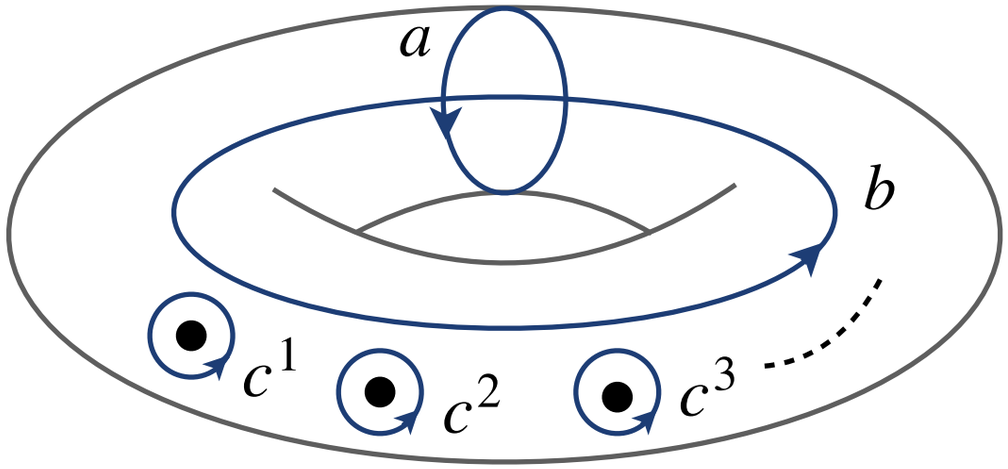
Wilson-’t Hooft lines as transfer matrices

Gauge theory side

From the 6d theory point of view, Wilson-’t Hooft lines in 4d come from codim-4 defects wrapping 1-cycles on the torus.

Consider a Wilson-’t Hooft line labeled by the 1-cycle:

$$\gamma_\sigma = b + \sum_r \frac{1-\sigma^r}{2} c^r$$



If $\sigma^r = +1$ (-1), the cycle passes above (below) the rth puncture. The cycle γ_σ determines the charge of the Wilson-’t Hooft line $\langle T_{(\mathbf{m},\mathbf{e})} \rangle$ in 4d :

$$\mathbf{m} = h_1^\vee \oplus \cdots \oplus h_1^\vee$$

$$\mathbf{e} \text{ specified by } \sigma \in \{\pm\}^n$$

Wilson-'t Hooft lines as transfer matrices

Gauge theory side

Put the circular quiver theory on 4d twisted spacetime $S^1 \times_\epsilon \mathbb{R}^2 \times \mathbb{R}$

and consider the Wilson-'t Hooft line $T_{(\mathbf{m},\mathbf{e})}$ wrapping the circle S^1,

whose vev can be computed by supersymmetric localization:

[Ito-Okuda-Taki]

$$\left\langle T_{(\mathbf{m},\mathbf{e})} \right\rangle = \sum_{i^1,\ldots,i^n} \prod_{r=1}^{n} e^{2\pi \mathrm{i} b^r_{i^r}} e^{\pi \mathrm{i} \sigma^r \left(a^{r+1}_{i^{r+1}} - a^r_{i^r}\right)} \ell_{m^r}\left(a^r, a^{r+1}\right)^{i^{r+1}}_{i^r}$$

$$\ell_m\left(a^r, a^s\right)^j_i = \left(\frac{\prod_{k(\neq i)} \sin \pi \left(a^r_k - a^s_j - m\right) \prod_{l(\neq j)} \sin \pi \left(a^r_i - a^s_l - m\right)}{\prod_{k(\neq i)} \sin \pi \left(a^r_{ki} - \frac{1}{2}\epsilon\right) \sin \pi \left(a^r_{ik} - \frac{1}{2}\epsilon\right)} \right)^{\frac{1}{2}}$$

where $\quad a^r = R(A^r_0 + i\phi^r_0)(\infty), \quad b^r = \dfrac{\Theta^r}{2\pi} - \dfrac{4\pi i R}{g^2}\phi^r_9(\infty)$

We can reproduce the same result by AGT, *i.e.* the Verlinde operator in Toda CFT.

[Alday-Gaiotto-Gukov-Tachikawa-Verlinde, Drukker-Gomis-Okuda-Teschner, Gomis-Le Floch]

Wilson-’t Hooft lines as transfer matrices

Correspondence

Compare

$$\langle T_{(\mathbf{m},\mathbf{e})}\rangle = \sum_{i^1,\ldots,i^n} \prod_{r=1}^{n} e^{2\pi \mathrm{i} b^r_{i^r}} e^{\pi \mathrm{i}\sigma^r (a^{r+1}_{i^{r+1}} - a^r_{i^r})} \ell_{m^r}(a^r, a^{r+1})^{i^{r+1}}_{i^r}$$

$$\mathcal{T}_{\sigma,m} = \sum_{i^1,\ldots,i^n} \left(\prod_{s=1}^{n} \Delta^s_{i_s}\right)^{\frac{1}{2}} \prod_{r=1}^{n} e^{\pi \mathrm{i}\sigma^r \left(a^{r+1}_{i^{r+1}} - a^r_{i^r}\right)} \ell_{m^r}\left(a^r, a^{r+1}\right)^{i^{r+1}}_{i^r} \left(\prod_{s=1}^{n} \Delta^s_{i_s}\right)^{\frac{1}{2}}$$

If we quantize a^r, b^r so that

$$\left[\hat{a}^r_i, \hat{b}^s_j\right] = -\mathrm{i}\frac{\epsilon}{2\pi}\delta^{rs}\left(\delta_{ij} - \frac{1}{N}\right)$$

then

$$\text{Weyl quantization of } \langle T_{(\mathbf{m},\mathbf{e})}\rangle = \mathcal{T}_{\sigma,m}$$

Wilson-'t Hooft lines as transfer matrices

Our claim is the correspondence : [Maruyoshi-TO-Yagi]

Gauge theory side *Lattice model side*

Wilson-'t Hooft lines = transfer matrices

- $\mathcal{N} = 2$ circular quiver theory on $S^1 \times \mathbb{R}^3$
- Wrap a Wilson-'t Hooft line T around S^1
- $\langle T \rangle$ is a function of Coulomb branch parameters
- Quantization of $\langle T \rangle$ coincides with the transfer matrix of trigonometric quantum integrable system
- <u>Related to other correspondences via string dualities</u>

Notice

The details might be too technical, you can ignore them.
Today's message is the above correspondence, that's all.

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  Wilson-’t Hooft line on $S^1 \times \mathbb{R}^3$ = Transfer matrix of trigonometric type

- ☐ Emergence of integrability from TQFT in extra dimensions

Today’s message

“Wilson-’t Hooft lines = transfer matrices”

The details might be too technical, you can ignore them.

Integrability from TQFT in extra dimensions

M-theory setup

11d spacetime	$\mathbb{R}_0$	$\mathbb{R}^2_{12}$	S^1_3	$\mathbb{R}^2_{45}$	S^1_6	$\mathbb{R}_7$	$\mathbb{R}_8$	$\mathbb{R}_9$	S^1_{10}
N M5	$\mathbb{R}_0$	$\mathbb{R}^2_{12}$	S^1_3	–	S^1_6	–	–	–	S^1_{10}
n M5'	$\mathbb{R}_0$	$\mathbb{R}^2_{12}$	S^1_3	–	–	–	$\mathbb{R}_8$	$\mathbb{R}_9$	–
M2	–	–	S^1_3	–	S^1_6	–	$\mathbb{R}^+_8$	–	–

N M5 : 6d $\mathcal{N} = (2,0)$ theory on $\mathbb{R}_0 \times \mathbb{R}^2_{12} \times_\epsilon S^1_3 \times S^1_6 \times S^1_{10}$

n M5' : n punctures on the torus $S^1_6 \times S^1_{10}$

M2 : Surface defect wrapping $S^1_3 \times S^1_6$

Reduction on $S^1_6 \times S^1_{10}$ gives

our 4d setup with a line defect wrapping the circle S^1_3

Integrability from TQFT in extra dimensions

Compactify $\mathbb{R}_9 \to S^1_9$, reduce on S^1_3, and apply T-duality $S^1_9 \to \check{S}^1_9$,
then we now have type IIB string theory setup:

10d spacetime	$\mathbb{R}_0$	$\mathbb{R}^2_{12}$	$\mathbb{R}^2_{45}$	S^1_6	$\mathbb{R}_7$	$\mathbb{R}_8$	$\check{S}^1_9$	S^1_{10}
N D5	$\mathbb{R}_0$	$\mathbb{R}^2_{12}$	$-$	S^1_6	$-$	$-$	$\check{S}^1_9$	S^1_{10}
n D3	$\mathbb{R}_0$	$\mathbb{R}^2_{12}$	$-$	$-$	$-$	$\mathbb{R}_8$	$-$	$-$
F1	$-$	$-$	$-$	S^1_6	$-$	$\mathbb{R}^+_8$	$-$	$-$

N D5 : 6d $\mathcal{N} = (1,1)$ SYM on $\mathbb{R}_0 \times \mathbb{R}^2_{12} \times S^1_6 \times \check{S}^1_9 \times S^1_{10}$
n D3 : codim-3 defects on $\mathbb{R}_0 \times \mathbb{R}^2_{12}$
F1 : Wilson line on S^1_6

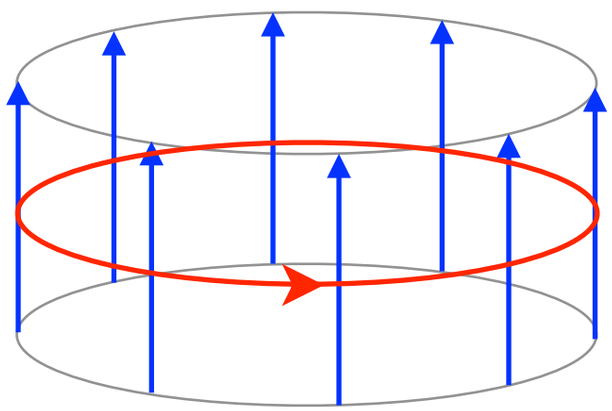
Integrability from TQFT in extra dimensions

Ω-deformation of 6d $\mathcal{N} = (1,1)$ SYM on $\mathbb{R}_0 \times \mathbb{R}^2_{12} \times S^1_6 \times \check{S}^1_9 \times S^1_{10}$

—> Costello's 4d Chern-Simons on $\mathbb{R}_0 \times S^1_6 \times \check{S}^1_9 \times S^1_{10}$ [Costello-Yagi]

n D3 on $\mathbb{R}_0 \times \mathbb{R}^2_{12}$ —> line defects on $\mathbb{R}_0$

F1 on S^1_6 —> Wilson line on S^1_6



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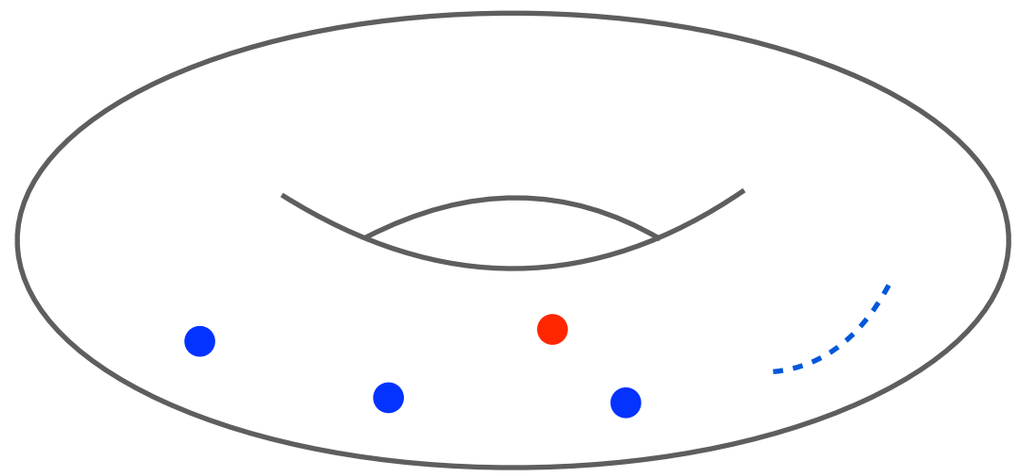

$\mathbb{R}_0 \times S^1_6$

$\check{S}^1_9 \times S^1_{10}$

Line operators in 4d CS generate integrable lattice models.

[Costello, Costello-Witten-Yamazaki]

Decompactify $S^1_9 \to \mathbb{R}_9$, this is our trigonometric limit.

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