KPZ equation, attractive bosons, and Efimov effect

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KPZ equation

Kardar, Parisi & Zhang, PRL (1986)

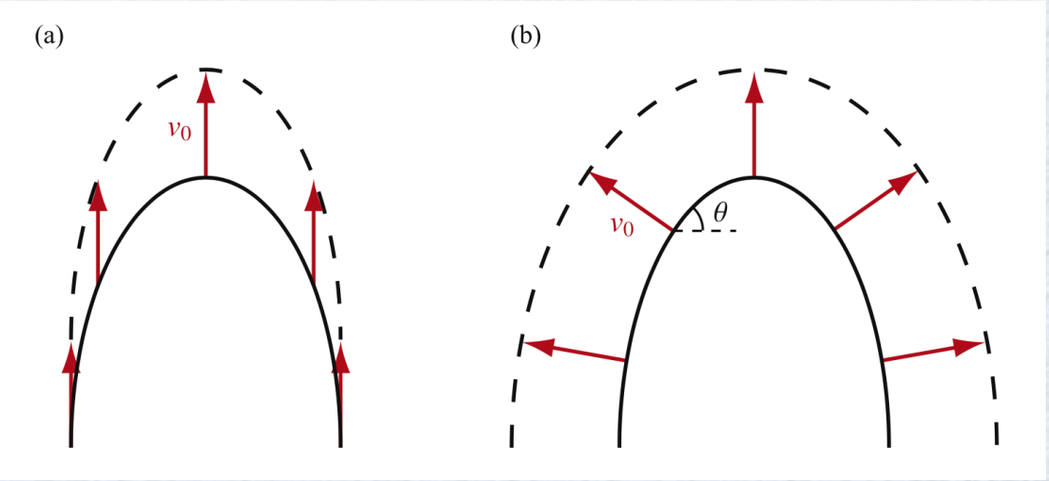
$$\frac{\partial h}{\partial t} = v + \nu \nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \eta$$

describes growth of d-dimensional surface $h(t, \vec{r})$

- v : **constant force (removed by $h \to h + vt$)**
- ν : **diffusion**
- η : **white Gaussian noise with zero mean**

$$\langle \eta(t, \vec{r}) \eta(t', \vec{r}') \rangle = D \delta(t - t') \delta(\vec{r} - \vec{r}')$$

- λ : **nonlinearity from geometric effect**



Takeuchi, Physica A (2018)

KPZ equation

Kardar, Parisi & Zhang, PRL (1986)

$$\frac{\partial h}{\partial t} = \nu \nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \eta$$

Roughness of surface is described by

$$\langle [h(t,\vec{r}) - h(0,\vec{0})]^2 \rangle \sim r^{2\chi} F\left(\frac{t}{r^z}\right)$$

- χ : **roughness exponent**

  Surface is rough for $\chi > 0$ and smooth for $\chi < 0$

- z : **dynamical exponent**

Exact values in 1D are $\chi = 1/2$ & $z = 3/2$

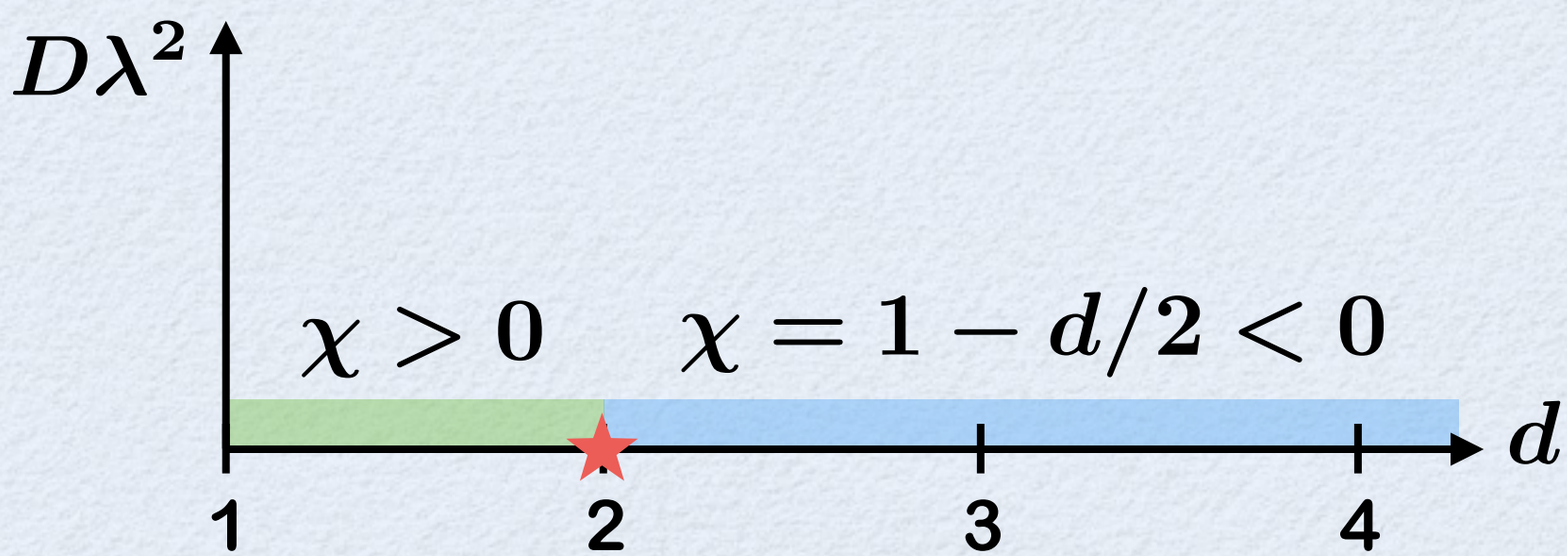
Huge development in theory, exp. & math in 1D

Sasamoto & Spohn (2010); Takeuchi & Sano (2010); Hairer (2013); ...

KPZ in higer dimensions

$$\frac{\partial h}{\partial t} = \nu \nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \eta$$

- For d < 2, λ is relevant → surface is rough
- For d > 2, λ is irrelevant → surface is smooth



Tang, Nattermann & Forrest PRL (1990)

KPZ in higer dimensions

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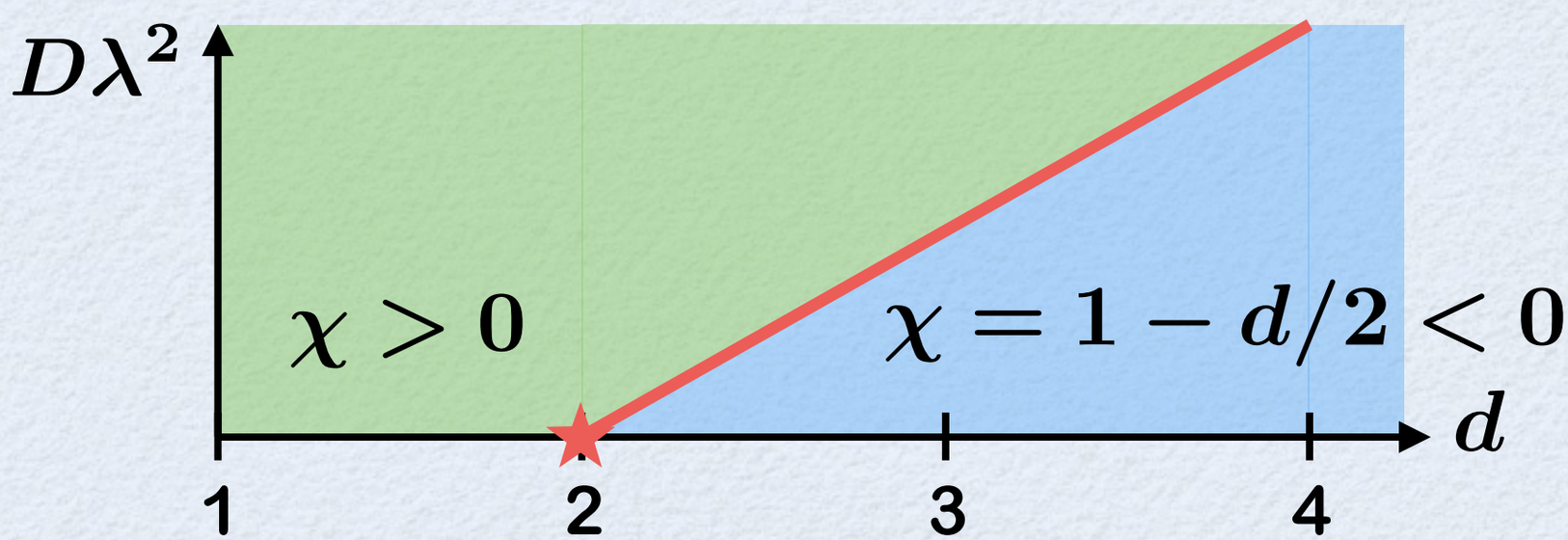
With increasing λ, there is a phase transition from smooth to rough phases, at which $\chi = 0$ & $z = 2$

→ Roughening transition

Non-relativistic CFT ?



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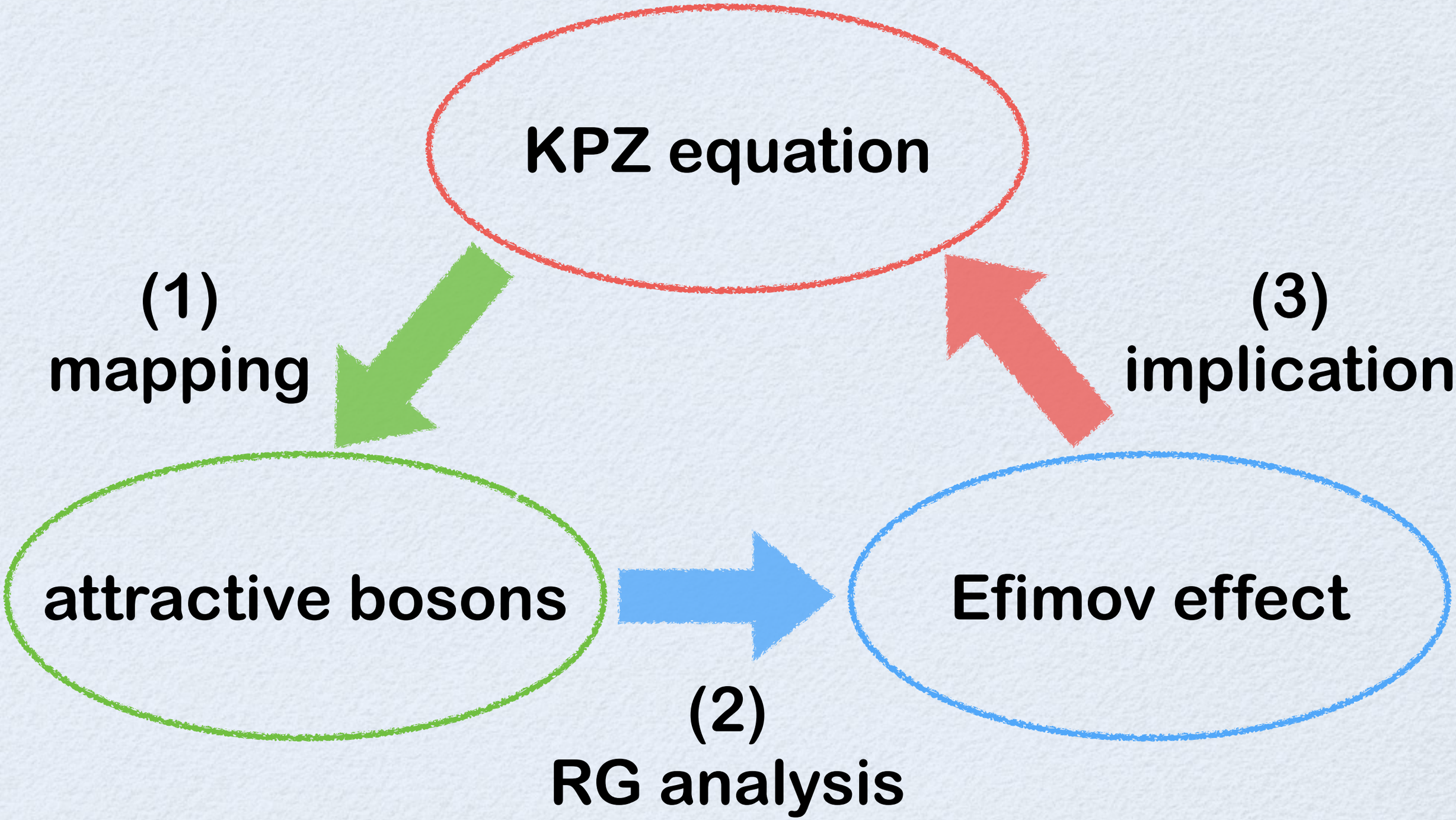
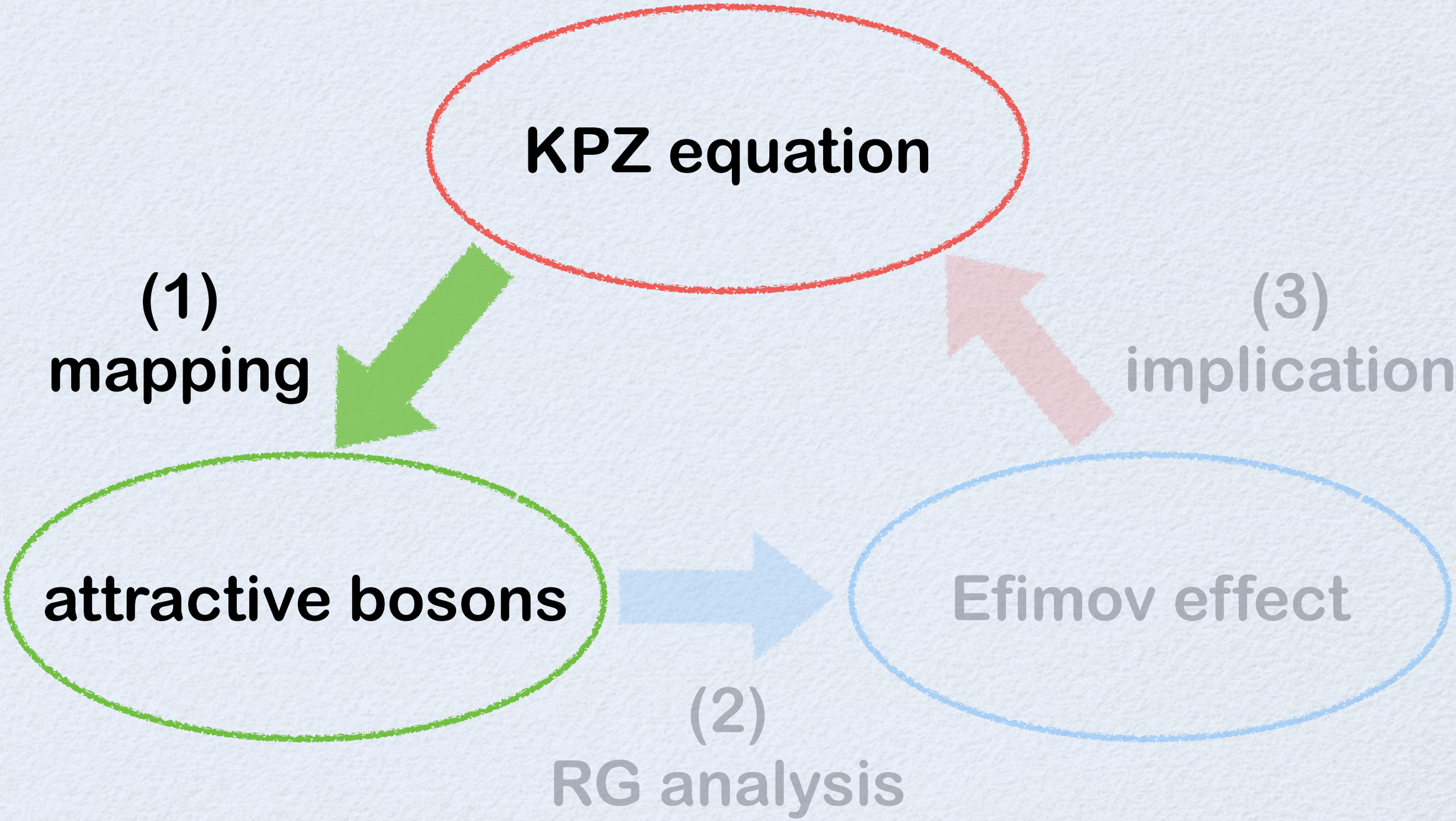
Plan of this talk



Ref: Y. Nanakaya & Y. Nishida, arXiv:2010.15161

"Efimov effect at the Kardar-Parisi-Zhang roughening transition"

Plan of this talk



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"Efimov effect at the Kardar-Parisi-Zhang roughening transition"

Field theoretical formation

Solve KPZ eq for a given η → h_η

Martin, Siggia & Rose (1973)
Janssen (1976)
De Dominicis (1976)

Ensemble average

$$\langle \mathcal{O}(h) \rangle = \int \mathcal{D}\eta \, \mathcal{O}(h_\eta) \, e^{-\frac{1}{2D}\int_{t,\vec{r}} \eta^2}$$

$$\mathcal{O}(h_\eta) = \int \mathcal{D}h \, \mathcal{O}(h) \prod_{t,\vec{r}} \delta\left[\frac{\partial h}{\partial t} - \nu \nabla^2 h - \frac{\lambda}{2}(\nabla h)^2 - \eta\right]$$

$$= \int \mathcal{D}h \mathcal{D}\bar{h} \, \mathcal{O}(h) \exp\left[-\int_{t,\vec{r}} i\bar{h}\left[\frac{\partial h}{\partial t} - \cdots - \eta\right]\right]$$

Gaussian integration over η

$$\langle \mathcal{O}(h) \rangle = \int \mathcal{D}h \mathcal{D}\bar{h} \, \mathcal{O}(h) \, e^{-S}$$

with "action"

$$S = \int_{t,\vec{r}} i\bar{h}\left[\frac{\partial h}{\partial t} - \nu \nabla^2 h - \frac{\lambda}{2}(\nabla h)^2 - \frac{D}{2} i\bar{h}\right]$$

Field theoretical formation

Cole-Hopf transformation $h = \frac{2\nu}{\lambda} \ln \phi \;\; \& \;\; i\bar{h} = \frac{\lambda}{2\nu} \bar{\phi}\phi$

$$\Rightarrow \quad S = \int_{\tau,\vec{r}} \left[\bar{\phi} \left(\frac{\partial}{\partial \tau} - \frac{\nabla^2}{2} \right) \phi - \frac{D\lambda^2}{16\nu^3} (\bar{\phi}\phi)^2 \right]$$

$\tau \equiv 2\nu t$

Attractive bosons with a delta-potential !

(attraction of bosons ~ nonlinearity of KPZ)

- **Bethe ansatz in 1D allows exact analyses**

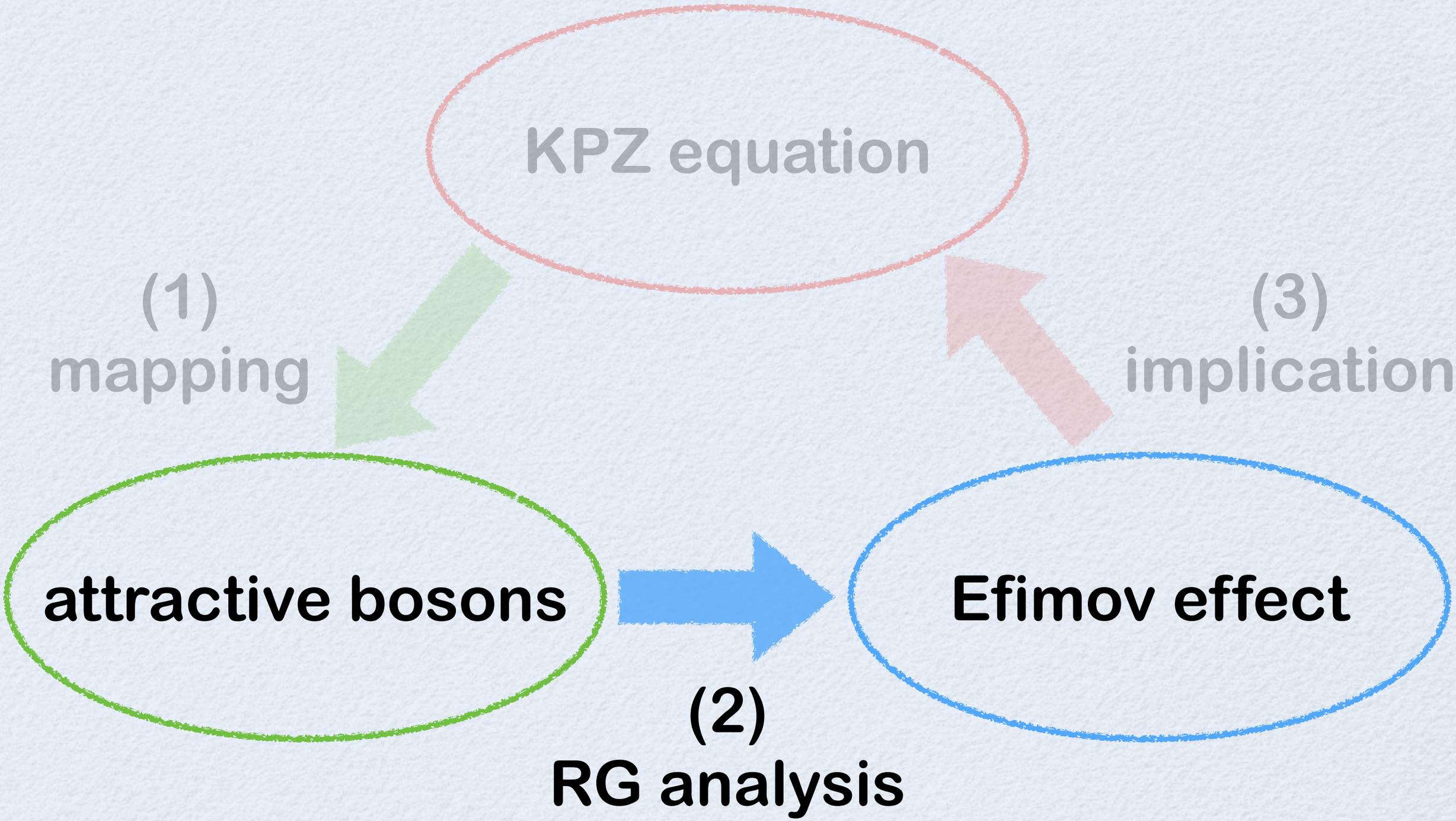
  Kardar (1987); Calabrese, Le Doussal & Rosso (2010); Dotsenko (2010); …

- **Delta-potential for d ≥ 2 requires regularization**

  (e.g. sharp momentum cutoff $|\vec{p}| < \Lambda$ **)**

  ⇒ **Continuum limit recovered @ critical point**

Plan of this talk



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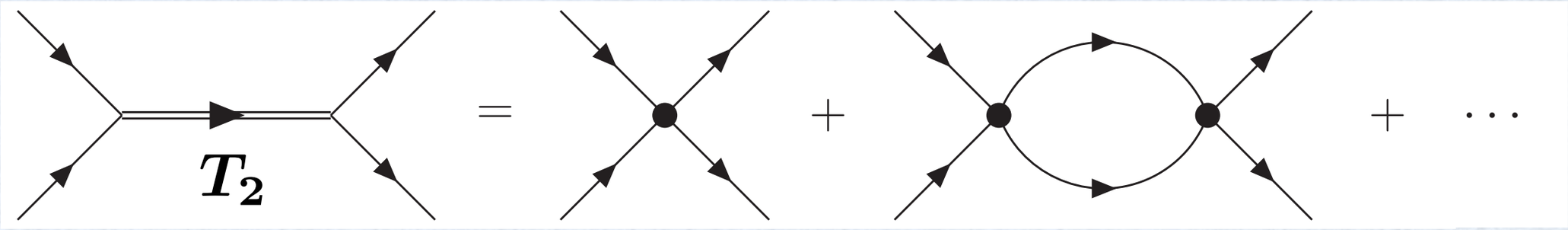
"Efimov effect at the Kardar-Parisi-Zhang roughening transition"

RG of 2-boson coupling

$$S_\Lambda = \int_{\tau,\vec{r}} \left[\bar{\phi} \left(\frac{\partial}{\partial \tau} - \frac{\nabla^2}{2} \right) \phi - \frac{g_2}{4} (\bar{\phi}\phi)^2 - \cdots \right]$$

2-boson scattering amplitude ($\hat{g}_2(\Lambda) \equiv \Lambda^{d-2} g_2$)



$$\frac{1}{T_2(k_0, \vec{k})} = \frac{\Lambda^{d-2}}{\hat{g}_2} - \frac{\Lambda^{d-2}}{(d-2)(4\pi)^{d/2}\Gamma(d/2)}$$

$$- \frac{\Gamma(1-d/2)}{2(4\pi)^{d/2}} \left(\frac{k^2}{4} - ik_0 \right)^{d/2-1} + O\left(\frac{k^2}{\Lambda^{4-d}} \right)$$

Callan-Symanzik eq $\left(\frac{\partial}{\partial \Lambda} + \frac{\partial \hat{g}_2}{\partial \Lambda} \frac{\partial}{\partial \hat{g}_2} \right) T_2 = 0$

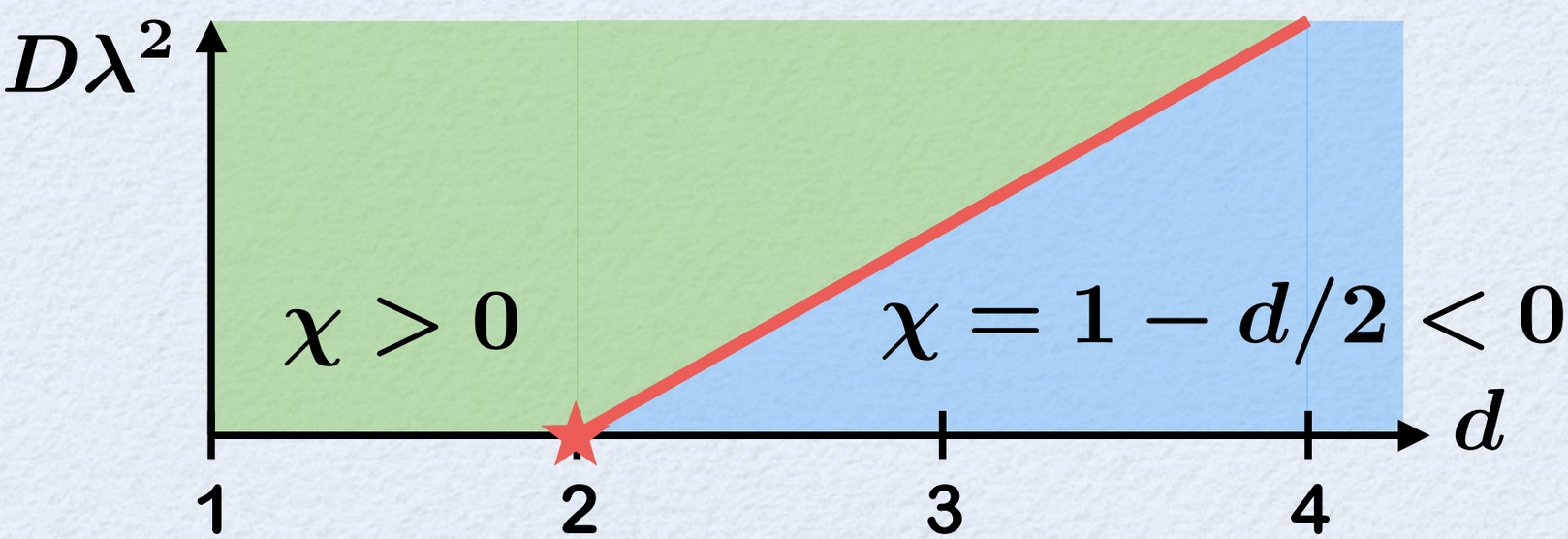
RG of 2-boson coupling

Beta function for 2 < d < 4 has two fixed points

$$\Lambda \frac{\partial \hat{g}_2}{\partial \Lambda} = (d-2)\hat{g}_2 - \frac{\hat{g}_2^2}{(4\pi)^{d/2}\Gamma(d/2)}$$

0 $\qquad\qquad$ $\hat{g}_2^* = (d-2)(4\pi)^{d/2}\Gamma(d/2)$

With increasing g_2, there is a binding transition from unbound to bound bosons



RG of 2-boson coupling

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With increasing g_2, there is a binding transition from unbound to bound bosons

KPZ roughening transition

2-boson scattering amplitude at the critical point

$$T_2(k_0, \vec{k})\Big|_{\hat{g}_2=\hat{g}_2^*} = -\frac{2(4\pi)^{d/2}}{\Gamma(1-d/2)}\left(\frac{k^2}{4} - ik_0\right)^{1-d/2}$$

$z = 2$

Interim summary

Attractive bosons

- unbound bosons

  binding transition

- bound bosons

KPZ equation

- smooth surface

  roughening transition

- rough surface

The critical point is described by non-relativistic CFT

$$S_\Lambda = \int_{\tau,\vec{r}} \left[\bar{\phi} \left(\frac{\partial}{\partial \tau} - \frac{\nabla^2}{2} \right) \phi - \frac{g_2^*}{4} (\bar{\phi}\phi)^2 - \cdots \right]$$

This is the end of story,

if all higher-order terms are irrelevant

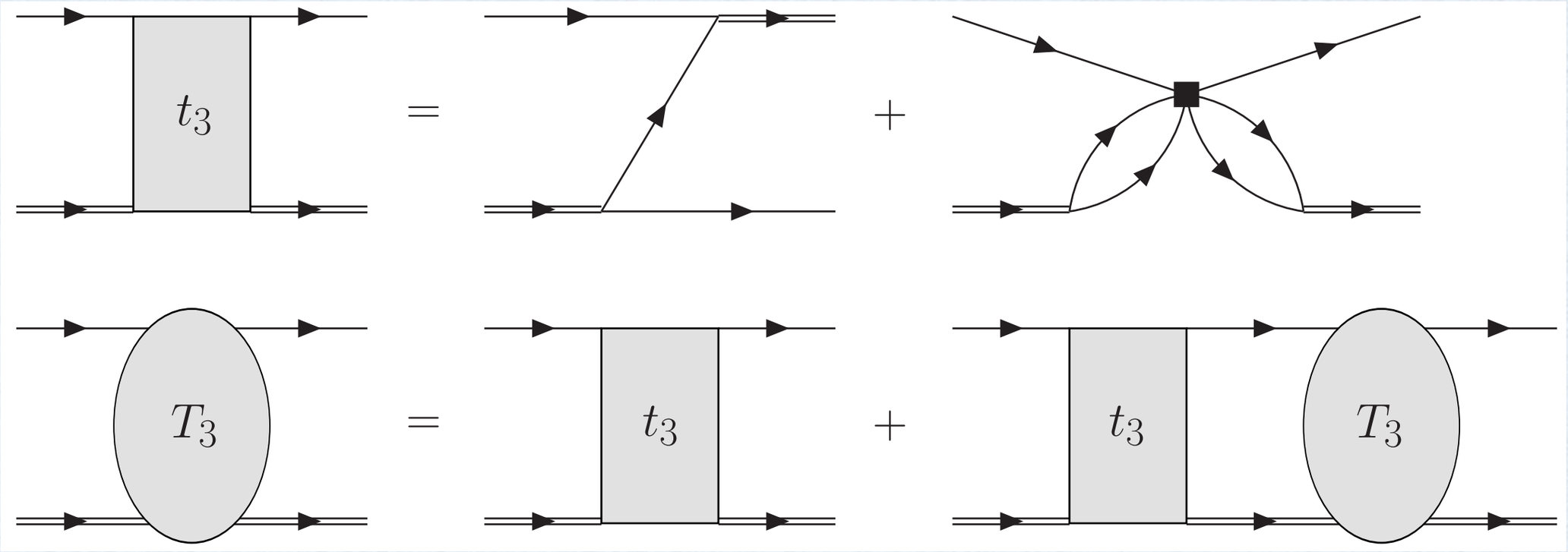
However, for bosons at the critical point in 3D,

3-boson coupling is relevant, leading to Efimov effect

RG of 3-boson coupling

$$S_\Lambda = \int_{\tau,\vec{r}} \left[\bar{\phi} \left(\frac{\partial}{\partial \tau} - \frac{\nabla^2}{2} \right) \phi - \frac{g_2^*}{4} (\bar{\phi}\phi)^2 - \frac{g_3}{36} (\bar{\phi}\phi)^3 - \cdots \right]$$

3-boson scattering amplitude ($\hat{g}_3(\Lambda) \equiv \Lambda^2 g_3/(9g_2^2)$)

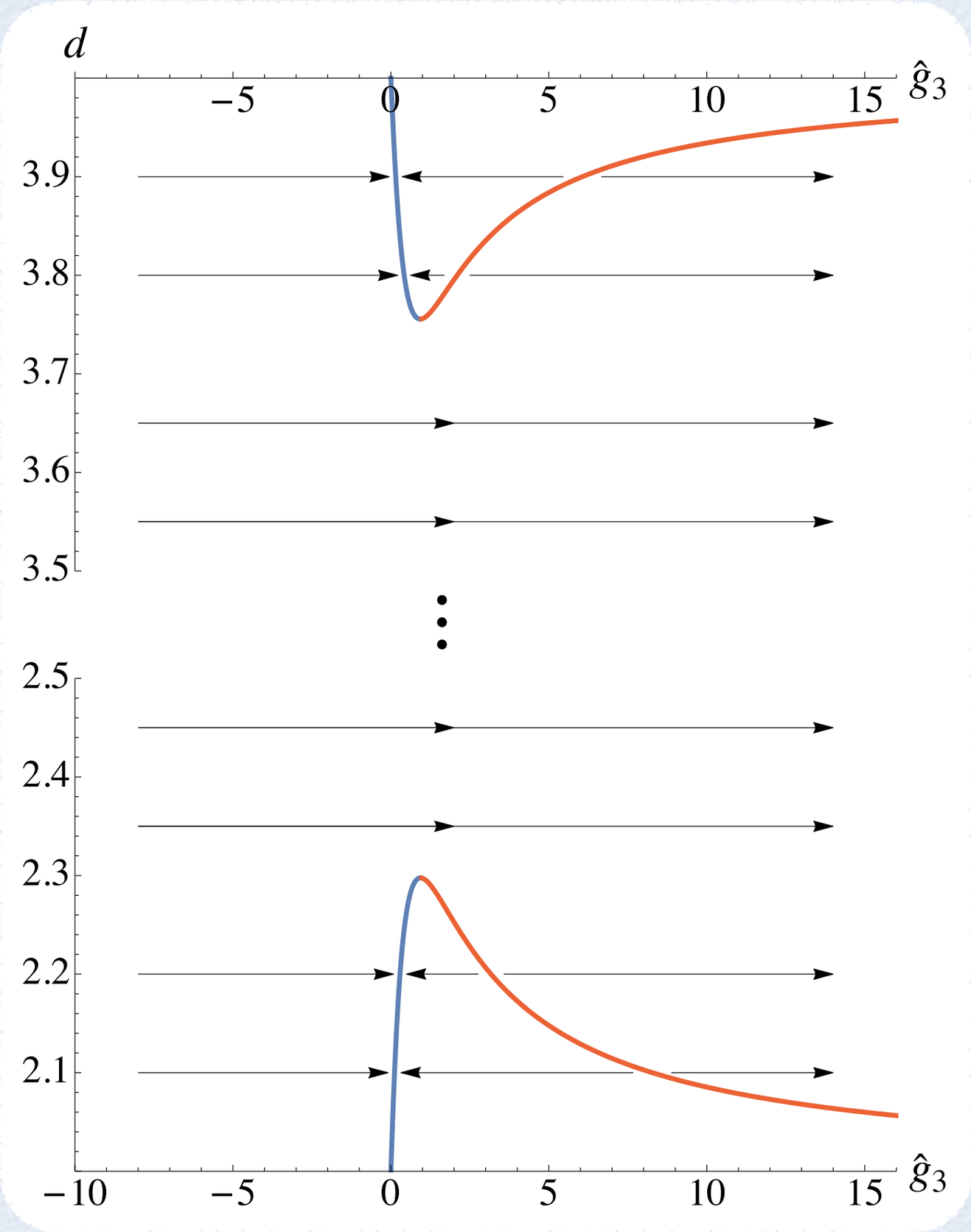


Bedaque, Hammer & van Kolck (1999)

STM integral equation needs to be solved numerically

Callan-Symanzik eq $$\left(\frac{\partial}{\partial \Lambda} + \frac{\partial \hat{g}_3}{\partial \Lambda} \frac{\partial}{\partial \hat{g}_3} \right) T_3 = 0$$

RG of 3-boson coupling



- For d < 2.30 & 3.76 < d, IR fixed point exists and the critical point is still described by NRCFT
- For 2.30 < d < 3.76, Fixed points disappear and scale inv. is lost

$$\hat{g}_3 = c\,\frac{1 - s_0 \tan(s_0 \ln \Lambda/\Lambda_*)}{1 + s_0 \tan(s_0 \ln \Lambda/\Lambda_*)}$$

Discrete scale invariance

$$\Lambda \to e^{-\pi n/s_0}\Lambda \quad (n \in \mathbb{Z})$$

Efimov effect

Efimov, PLB (1970); NPA (1973)

For 3 bosons at the critical point in 3D,

$$S_\Lambda = \int_{\tau,\vec{r}} \left[\bar{\phi} \left(\frac{\partial}{\partial \tau} - \frac{\nabla^2}{2} \right) \phi - \frac{g_2^*}{4} (\bar{\phi}\phi)^2 - \frac{g_3}{36} (\bar{\phi}\phi)^3 - \cdots \right]$$

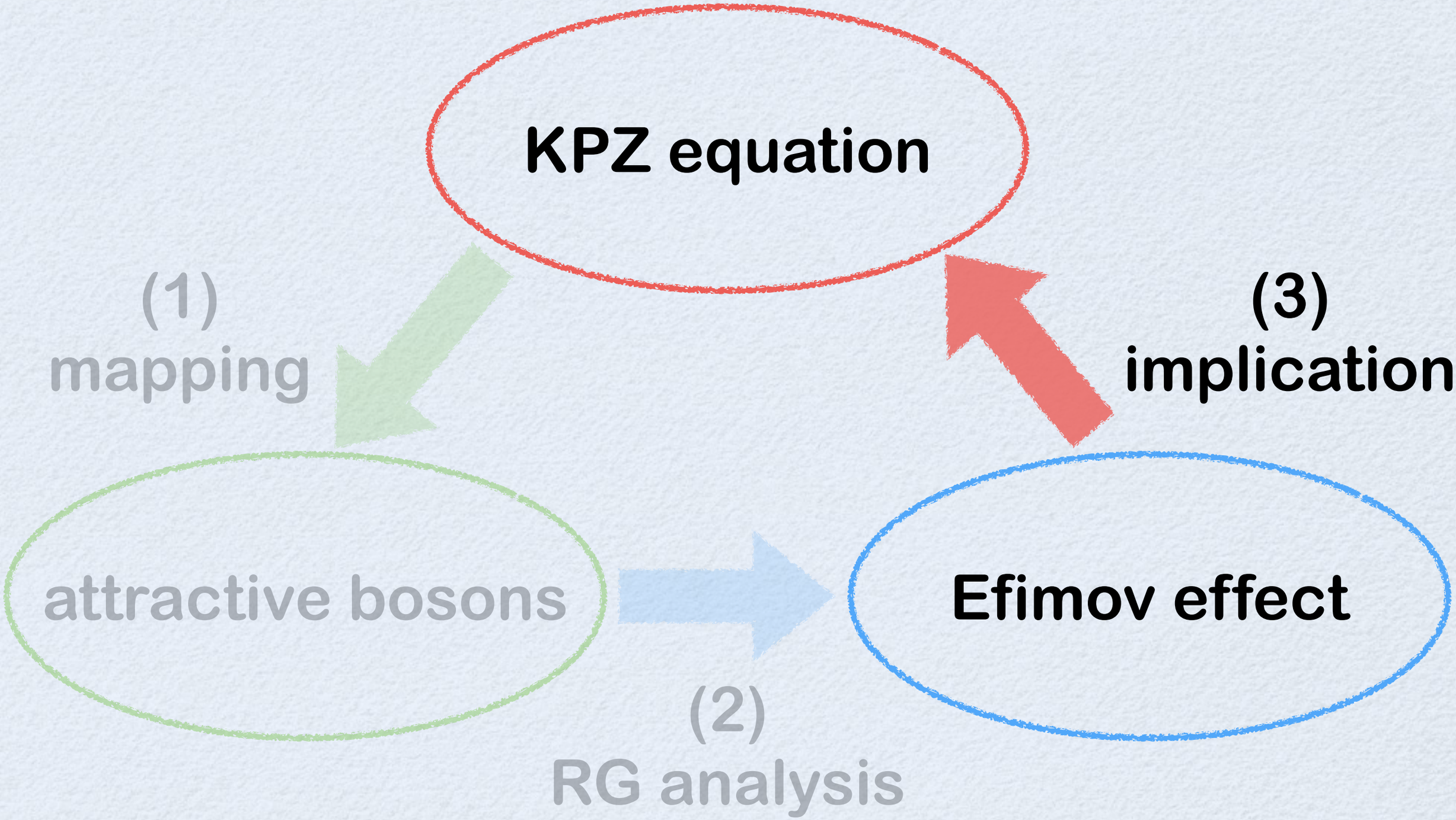
scale invariant under z = 2 is broken by running of g_3 down to discrete scale invariance with $e^{\pi/s_0} \approx 22.7$

This discrete scale invariance survives for 4 bosons, but unestablished for more bosons

Deltuva (2010-2013)

- If all higher-order terms are irrelevant, DSI persists for any number of bosons
- If some higher-order term is relevant, no scale invariance remains

Plan of this talk



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"Efimov effect at the Kardar-Parisi-Zhang roughening transition"

Our prediction

Roughening transition governed by KPZ equation

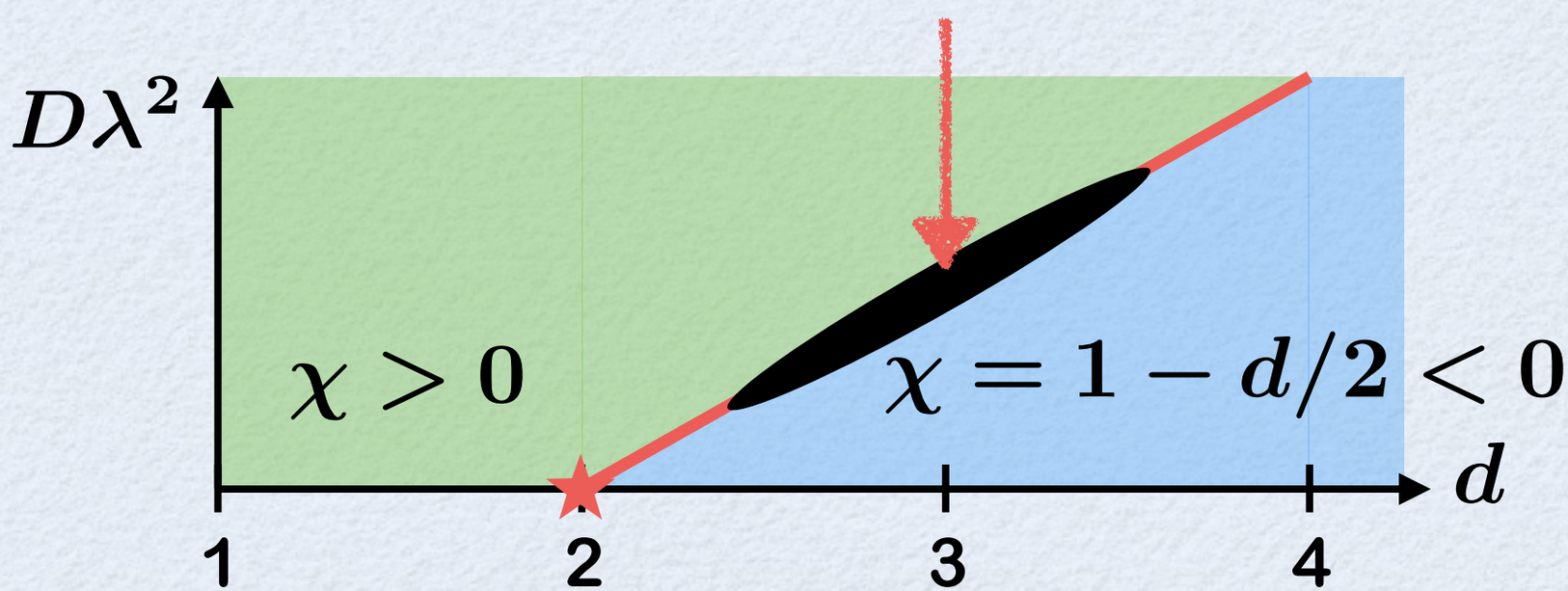
Binding transition of bosons → Efimov effect

Because full scale invariance under z = 2 is lost,

$$\langle [h(t,\vec{r}) - h(0,\vec{0})]^2 \rangle \sim r^{2\chi} F\left(\frac{t}{r^z}\right)$$

with $\chi = 0$ & $z = 2$ is no longer expected

but some unusual behavior is expected to emerge



Our prediction

- If discrete scale invariance persists

$$\langle [h(t,\vec{r}) - h(0,\vec{0})]^2 \rangle \sim f_{s_0}[\ln(\Lambda_* r)]\, F\left(\frac{t}{r^2}\right)$$

$$f_{s_0}[\ln(\Lambda_* r)] = f_{s_0}[\ln(\Lambda_* r) + \pi/s_0]$$

DSI under $\vec{r} \to (22.7)^n \vec{r}, \quad t \to (22.7)^{2n} t$

Efimov effect emerges at KPZ roughening transition !

- If no scale invariance remains

Discontinuous (1st-order) transition

Our prediction is speculative, which should be proven/disproven by numerical simulations or more rigorous mathematical approaches