Conserved charges in gravity and entropy

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References:

S. Aoki, T. Onogi and S. Yokoyama,
"Conserved charge in general relativity",
arXiv:2005.13233[gr-qc].

S. Aoki, T. Onogi and S. Yokoyama,
"Charge conservation, Entropy, and Gravitation",
arXiv:2010.07660[gr-qc].

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Tetsuya Onogi

Shuichi Yokoyama

Motivation

What is energy or energy density in curved spacetime ?

Why do we care ?

Flow equation

AdS/CFT from boundary QFT

CFT → AdS

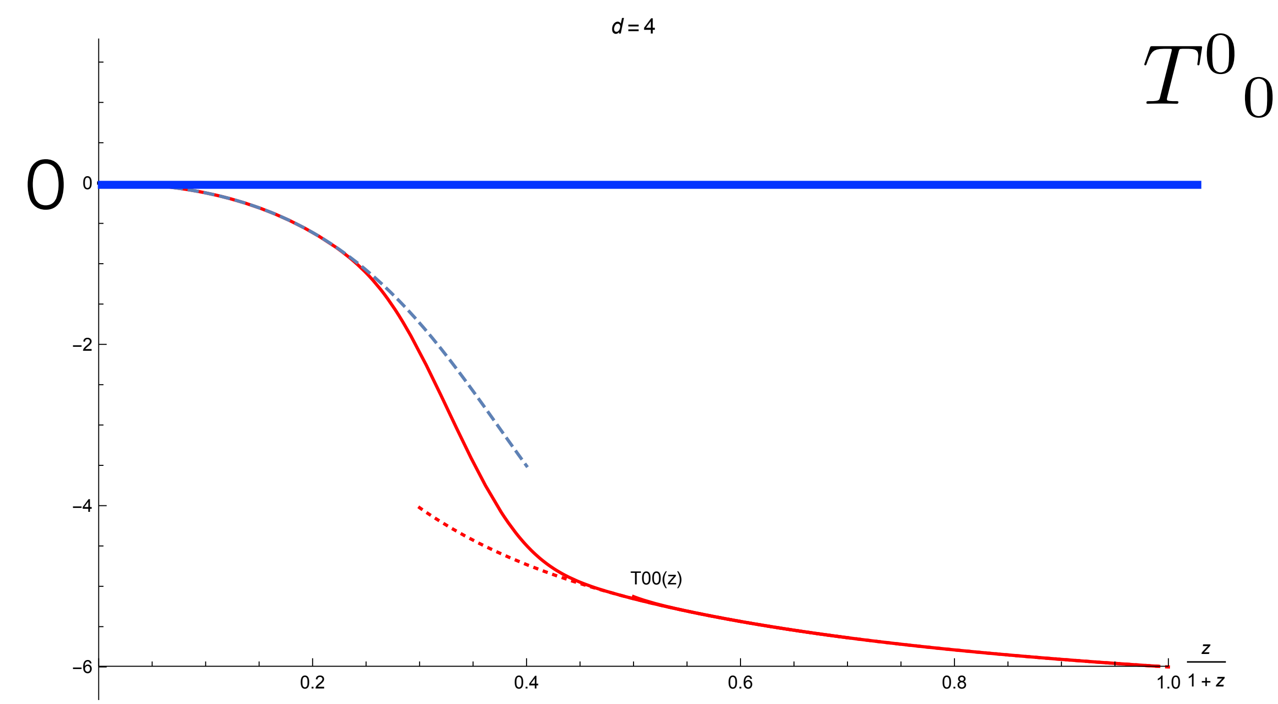
CFT@ finite T → Black hole in AdS ?

arXiv:2004.03779, S. Aoki, T. Onogi, S. Yokoyama
" What does a quantum black hole look like ?"

massless free scalar @ finite T → Black hole-like object + matter in AdS

energy distribution on curved spacetime ?

$$T_{00},\ T^{00},\ T^{0}{}_{0},\ \sqrt{-g}T_{00},\ \cdots\ ?$$



O. Energy in general relativity

(Conserved) energy in general relativity

Einstein equation $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G_d T_{\mu\nu}$ $\quad T_{\mu\nu}(x) = \frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}(x)}$

gravity **matter**

Bianchi identity

conservation $\nabla^\mu T_{\mu\nu} = 0$ but what we need for a conservation is $\partial^\mu T_{\mu\nu} \neq 0$

rewrite

$$\partial^\mu \left[\sqrt{|g|} \left(T_{\mu\nu} + \underline{t_{\mu\nu}} \right) \right] = 0$$

Einstein's pseudo-tensor

gravitational energy ?

This method works only for asymptotically flat $\Lambda = 0,\ T_{\mu\nu} \to 0\ (r \to \infty)$

in Cartesian coordinate (not in polar coordinate)

for two or more particles

$t_{\mu\nu}$ is not covariant under general coordinate transformation.

violate the fundamental principle of general relativity !

Quasi-local energy

Arnowitt-Deser-Misner (ADM) energy

Komar energy, Bondi energy

Hamiltonian with Gibbons-Hawking term

$$E = \int dV \, (\text{local energy})$$

$$E = \int_{r\to\infty} dS \, (\text{quasi-local energy})$$

cf. Gauss's law in electromagnetism

$$Q = \int_V dV \, J_0 = \int_{\partial V} dS_\mu F^{0\mu}$$

ex. Komar energy **Komar, PR127(1962) 1411**

$$E(\xi) = \frac{c}{16\pi G_d} \int_{\Sigma(x_0)} d\Sigma_0 \sqrt{-g}\, \nabla_\nu \nabla^{[0}\xi^{\nu]} = \frac{c}{16\pi G_d} \int_{\partial\Sigma(x_0)} d\Sigma_{0k} \sqrt{-g}\, \nabla^{[0}\xi^{k]}$$

Quasi-local energy

c: some constant

This is a Noether charge of the 2nd type for a coordinate transformation ξ^ν.

For $\xi^\mu = -\delta^\mu_0$, **Killing vector**

$$E = -\frac{2c}{16\pi G_d} \int_{\Sigma(x_0)} d\Sigma_0 \sqrt{-g}\, R^0{}_0$$

However quasi-local energy cannot tell a distribution of energy.

Local energy must exist since quasi-local energy is derived from it.

$\because$ local energy (mass) is a source which generates gravitational fields.

The covariant definition for local energy is a missing piece in general relativity.

Our aim

To give a precise and universal definition of energy by the volume integral of local energy if exists and extend it to more general cases.

Part I. Conserved charges with symmetry (in the presence of Killing vector)

Energy.

Part II. Conserved charges without symmetry

Generic conserved charge in GR. meaning ?

Part I.
Conserved charges with symmetry

S. Aoki, T. Onogi and S. Yokoyama,
"Conserved charge in general relativity",
arXiv:2005.13233[gr-qc]

Symmetry → **Killing vector** $\mathcal{L}_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$

→ **covariantly conserved vector current**

$$\nabla_\mu (T^\mu{}_\nu \xi^\nu) = (\nabla_\mu T^\mu{}_\nu)\xi^\nu + \frac{1}{2} T^{\mu\nu}(\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu) = 0$$

→ $$\boxed{Q(\xi) = \int_{\Sigma(x_0)} d\Sigma_0 \sqrt{-g}\, T^0{}_\nu \xi^\nu}$$ **conserved charge**

scalar

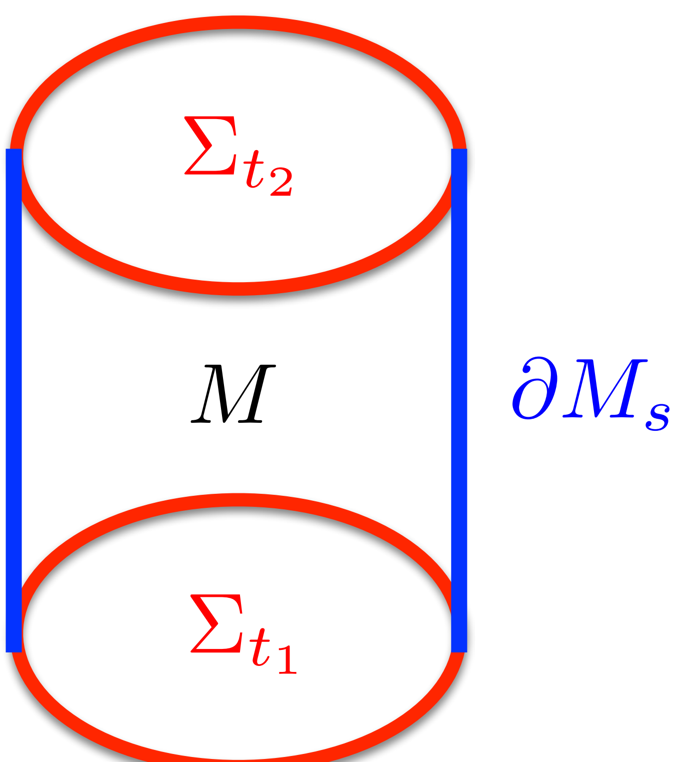
$$\because \quad 0 = \int_M d^d x \sqrt{|g|}\, \nabla_\mu J^\mu = \int_M d^d x\, \partial_\mu \left(\sqrt{|g|} J^\mu \right) = \int_{\partial M} d\Sigma_\mu \sqrt{|g|}\, J^\mu$$

$J^\mu := T^\mu{}_\nu \xi^\nu$

$$\nabla_\mu J^\mu = \frac{1}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|} J^\mu \right)$$

Stokes' theorem

$$\partial M = \partial M_s \oplus \Sigma_{t_2} \ominus \Sigma_{t_1}$$



assume

$d\Sigma_k J^k = 0$ on ∂M_s → $Q(\Sigma_{t_2}) = Q(\Sigma_{t_1})$

ex. stationary space time $\longrightarrow$ **Killing vector** $\xi^\mu = -\delta^\mu_0$

a metric $g_{\mu\nu}$ does not contain x^0

conserved energy $\boxed{E = -\int_{\Sigma(x_0)} d\Sigma_0 \sqrt{-g}\, T^0{}_0}$

Covariant and universal definition of total energy

works for an arbitrary asymptotic behavior in an any coordinate system.

works for dynamical as well as back ground metric.

cf. Quasi-local energy

M. Shibata, "Numerical relativity" (100 Years of General Relativity-Vol. 1, World Scientific)

ADM mass

The ADM mass is not defined in a covariant way,

For its definition to be valid, in addition to asymptotic flatness, the metric components have to approach those of the flat Minkowski metric sufficiently quickly

A well-known example in which the ADM mass cannot be calculated is the metric of a four-dimensional non-spinning black hole in Painleve-Gullstrand coordinate,

Komar mass

this term does not denote the mass of the black hole in general (e.g., in the presence of a tours surrounding the black hole).

Our definition has been known, but rarely used.

1. V. Fock, *TheTheory of Space, Time and Gravitation* (Pergamon Press, New York 1959)

   The quantity $I = \int T^{\mu 0} \varphi_\mu \sqrt{-g}\, dx_1 dx_2 dx_3$ will be constant, $\cdots$, if the vector φ_μ satisfies the equation $\nabla_\nu \varphi_\mu + \nabla_\mu \varphi_\nu = 0$.

2. A. Tautman, Kings Collage lecture notes on general relativity, mimeographed note (unpublished), May-June 1958; Gen. Res. Grav. **34** (2002), 721-762, cited Fock.

3. A. Tautman's lecture notes was cited by Komar in PRD127(1962)1411.

These were forgotten in major textbooks (e.g. Landau-Lifshitz) except a few.

4. R. Wald, *General Relativity* (The University of Chicago Press, Chicago, 1984), p.286, footnote 3.

   a Killing vector field ξ^a is presented, $\cdots$, $\nabla^a(T_{ab}\xi^b) = (\nabla^a T_{ab})\xi^b + T_{ab}\nabla^a\xi^b = 0$, so $\int_\Sigma T_{ab}\xi^b n^a$ is conserved, i.e., independent of choice of Cauchy surface Σ.

   measure term is not specified, though.

See also lecture notes by Blau; Shiromizu (Japanese); Sekiguchi (Japanese).

No applications. **Let us consider some applications.**

1. Black holes

S. Aoki, T. Onogi and S. Yokoyama,
“Conserved charge in general relativity”,
arXiv:2005.13233[gr-qc]

1-1. (charged) Schwarzschild black hole

metric $$ds^2 = -(1+u)(dx^0)^2 - 2u dx^0 dr + (1-u)dr^2 + \underline{r^2 \bar{g}_{ij} dx^i dx^j}$$ (d-2)-sphere

Eddington-Finkelstein coordinate

$$u(r) = -\frac{r_0^{d-3}}{r^{d-3}} + \frac{Q^2}{r^{2(d-3)}} - \frac{2\Lambda r^2}{(d-2)(d-1)} \qquad Q^2 = 8\pi G_d \frac{(d-3)}{(d-2)} q^2$$

Energy Momentum tensor

$$T^0{}_0 = \frac{(d-2)}{16\pi G_d} \frac{\partial_r (r_0^{d-3} \delta u)}{r^{d-2}} = T^r{}_r \qquad T^i{}_j = \frac{\delta^i_j}{16\pi G_d} \frac{\partial_{rr}(r^{d-3}\delta u)}{r^{d-2}} \qquad \delta u(r) = -\frac{r_0^{d-3}}{r^{d-3}} + \frac{Q^2}{r^{2(d-3)}}$$

Energy of neutral black hole (Q=0) **Killing** $\xi^\mu = -\delta^\mu_0$

$$E = -\int_{x^0:\text{const.}} d^{d-1}x \sqrt{-g}\, T^0{}_0 = \frac{(d-2)V_{d-2}}{16\pi G_d} \int_0^\infty dr\, \partial_r (r^{d-3}\delta u(r)) = \boxed{\frac{(d-2)V_{d-2} r_0^{d-3}}{16\pi G_d}}$$

volume of (d-2)-sphere $V_{d-2} := \int d^{d-2}x \sqrt{\det \bar{g}_{ij}}$

This reproduces known results. For example $E = \dfrac{r_0}{2G_4} = M$ at $d=4$

Remark

$x^0 =$ constant hypersurface is space-like even inside the horizon.

However the Killing vector becomes space-like inside the horizon.

The BH energy is independent of the cosmological constant.

EMT for black hole

$$T^0{}_0 = T^r{}_r = -\frac{(d-2)r_0^{d-3}}{16\pi G_d}\frac{\delta(r)}{r^{d-2}} \qquad T^i{}_j = -\delta^i_j \frac{r_0^{d-3}}{16\pi G_d}\frac{r\delta^{(1)}(r)}{r^{d-2}}$$

See also Balasin-Nachbagauer, Class. Quant. Graf. 10(1993)2771.

Neutral black hole is NOT a vacuum solution to Einstein equation.

cf. Coulomb potential generated by a point charge is NOT a vacuum solution to Maxwell equation.

$$\nabla^2\left(\frac{1}{r^{d-3}}\right) = 0 \quad r \neq 0 \qquad \Rightarrow \qquad \nabla^2\left(\frac{1}{r^{d-3}}\right) \propto \delta(r)$$

Kerr BH See Balasin-Nachbagauer, Class. Quant. Graf. 11(1994)1453.

cf. Komar energy

$$E_{\text{Komar}} = \frac{c}{16\pi G_d}\int_{r\to\infty} d^{d-2}x\,\sqrt{-g}\,\nabla^{[0}\xi^{r]} = \lim_{r\to\infty}\frac{cV_{d-2}}{16\pi G_d}\left[(d-3)r_0^{d-3} - \frac{4\Lambda r^{d-1}}{(d-2)(d-1)}\right]$$

Komar energy for BH diverges for non-zero cosmological constant.

$$E_{\text{Komar}} = \frac{c(d-3)V_{d-2}r_0^{d-3}}{16\pi G_d} \quad \text{at } \Lambda = 0$$

our result

$$E = \frac{(d-2)V_{d-2}r_0^{d-3}}{16\pi G_d}$$

$$c = \frac{d-2}{d-3}$$

Both agree except d=3.

Our definition of energy is much more robust and universal.

Energy of charged black hole

$$E_Q = -\frac{(d-2)V_{d-2}}{16\pi G_d}\int_0^\infty dr\,\partial_r(r^{d-3}\delta u) \qquad \delta u(r) = -\frac{r_0^{d-3}}{r^{d-3}} + \frac{Q^2}{r^{2(d-3)}}$$

$$= E_0 - \frac{(d-3)V_{d-2}}{2}\int_0^\infty dr\,\partial_r\left(\frac{q^2}{r^{d-3}}\right) \longrightarrow \infty$$

Energy of a charged black hole diverges at d > 3.

This corresponds to a divergent self-energy of a charged point particle in the flat limit.

cf. Komar energy

$$\frac{c}{16\pi G_d}\int_{\Sigma(x^0)} d\Sigma_0\sqrt{-g}\nabla_\nu\nabla^{[0}\xi^{\nu]} = E_{\text{Komar}}(q=0) + \infty$$

Failure of Stokes' theorem

$$\frac{c}{16\pi G_d}\int_{\partial\Sigma(x^0)} d\Sigma_{0k}\sqrt{-g}\nabla^{[0}\xi^{k]} = E_{\text{Komar}}(q=0)$$

1-2. BTZ black hole

Bandaos-Teitelboim-Zanelli, PRL69(1992)1849.

d=3 AdS

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\phi - \omega(r)dt)^2$$

$$f(r) = \frac{r^2}{L^2} - m\theta(r) + \frac{J^2}{4r^2}, \quad \omega(r) = \frac{J}{2r^2}$$

L: AdS radius

Killing vectors $\xi_T^\mu := -\delta_0^\mu, \ \xi_S := \delta_\phi^\mu$

EMT

$$T^0_\nu = -\frac{\delta(r)}{16\pi G_3 r}\left(-m\delta^0_\nu + J\delta^\phi_\nu\right)$$

Energy

$$E = -\int_0^{2\pi} d\phi \int r dr\, T^0{}_0 = \frac{m}{8G_3}$$

$$E_{\mathrm{Komar}} = \infty$$

Angular momentum

$$P_\phi = \int_0^{2\pi} d\phi \int r dr\, T^0{}_\phi = \frac{J}{8G_3}$$

$$P_{\mathrm{Komar}} = \frac{cJ}{8G_3}$$

2. Compact star

S. Aoki, T. Onogi and S. Yokoyama,
"Conserved charge in general relativity",
arXiv:2005.13233[gr-qc]

Oppenheimer-Volkoff equation

stationary spherically symmetric metric

Oppenheimer-Volkoff, PR55(1939)374.

$$ds^2 = -f(r)(dx^0)^2 + h(r)dr^2 + r^2\tilde{g}_{ij}dx^i dx^j$$

with perfect fluid $\quad T^0{}_0 = -\rho(r), \quad T^r{}_r = P(r), \quad T^i_j = \delta^i_j P(r)$

Einstein equation $\Rightarrow$ Oppenheimer-Volkoff equation

$$-\frac{dP(r)}{dr} = \frac{G_d M(r)}{r^{d-2}}\left(P(r)+\rho(r)\right)h(r)\left\{d-3+\frac{r^{d-1}}{(d-2)M(r)}\left(8\pi P(r) - \frac{2\Lambda}{(d-1)G_d}\right)\right\}$$

with $\quad \dfrac{1}{h(r)} = k - \dfrac{2G_d M(r)}{r^{d-3}} - \dfrac{2\Lambda r^2}{(d-2)(d-1)}$

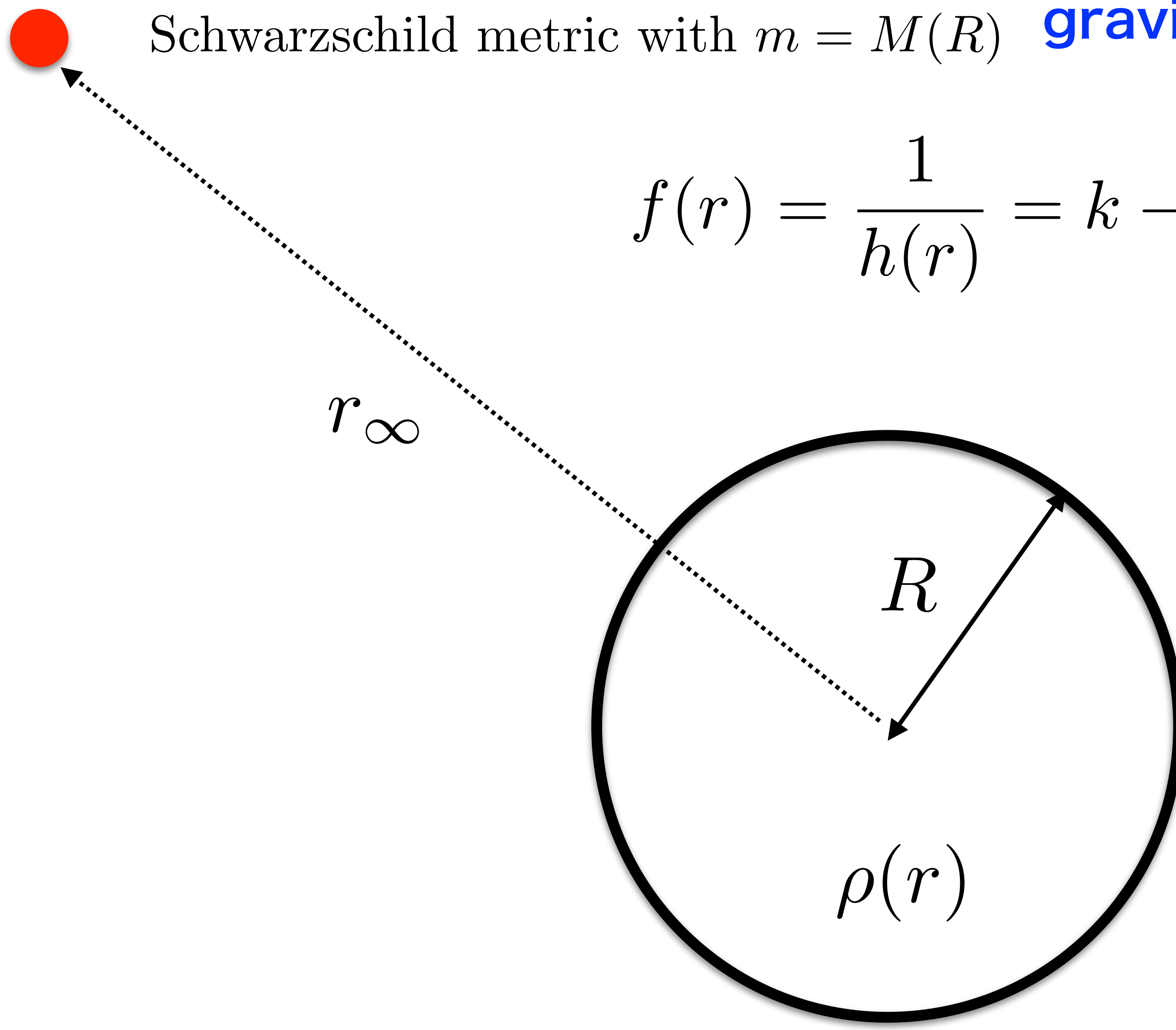
$$M(r) = \frac{8\pi}{d-2}\int_0^r ds\, s^{d-2}\rho(s), \quad M(0)=0$$

EOS $\quad P = P(\rho) \Rightarrow$ solution to OV equation

Schwarzschild metric with $m = M(R)$ **gravitational mass**

$$f(r) = \frac{1}{h(r)} = k - \frac{2G_d M(R)}{r^{d-3}} - \frac{2\Lambda r^2}{(d-2)(d-1)}$$

Schwarzschild metric



outside star

$$r > R, \quad \rho(r) = P(r) = 0$$

radius of compact star R from $P(r=R) = 0$

Energy of a compact star

conserved energy Killing vector $\xi^\mu = -\delta^\mu_0$

$$E = -\int d^{d-2}x \int_0^\infty dr \sqrt{|g|} T^0{}_0 = V_{d-2} \int_0^R \sqrt{f(r)h(r)} r^{d-2} \rho(r)$$

gravitational mass $M(R) = \frac{8\pi}{d-2} \int_0^R dr\, r^{d-2} \rho(r)$

extra factor $\sqrt{f(r)h(r)} \neq 1$ Schutz's textbook (guess)

Angus-Cho-Park, Eur. Phys. JC 78 (2018)500 (calculation)

$\sqrt{h(r)}$ incorrect guess in Wald's textbook

$$\frac{dM(r)}{dr} = \frac{8\pi}{d-2} r^{d-2} \rho(r)$$

$$E = \frac{(d-2)V_{d-2}}{8\pi} M(R) - G_d V_{d-2} \int_0^R dr \sqrt{f(r)h^3(r)}\, r M(r) \{\rho(r) + P(r)\}$$

gravitational mass corrections due to a structure inside star $:= \Delta E$

Physical meaning of ΔE

Newtonian limit

$$\Delta E \simeq -G_d V_{d-2} \int_0^R dr\, r M(r)\rho(r) + \cdots$$

gravitational interaction energy at d=4

$$U_4 := -\frac{G_d}{2}\int d^{d-1}x\, d^{d-1}y\, \frac{\rho(\vec{x})\rho(\vec{y})}{|\vec{x}-\vec{y}|} = -4\pi G_4 \int_0^R dr\, r\, M(r)\rho(r)$$

correction term represents the gravitational interaction energy !

cf. Komar energy

$$E_{\mathrm{Komar}} = \frac{c(d-3)V_{d-2}}{8\pi} M(R)$$

Komar energy misses the gravitational interaction energy.

$\boxed{\text{Our definition is physically more sensible.}}$

Size of gravitational interaction energy

constant density $\rho(r) = \rho_0$ $d = 4, \Lambda = 0, k = 0$

$$M_{\rm BH} = \frac{4\pi R^3 \rho_0}{3} \quad \text{fixed} \quad \Rightarrow \quad R \geq R_{\rm min} = \frac{9 G M_{\rm BH}}{4}$$

$$\Rightarrow \quad E = M_{\rm BH} \underbrace{- \pi\rho_0 \left[R(3r_0^2 - R^2) - 3r_0^2 \sqrt{r_0^2 - R^2} \sin^{-1}\left(\frac{R}{r_0}\right)\right]}_{\text{interaction energy}} \qquad r_0^2 := \frac{3}{8\pi G_4 \rho_0}$$

interaction energy $\simeq -68\%$ of $M_{\rm BH}$ at $R = R_{\rm min}$

A total energy is 1/3 of the observed mass.

Gravitational interaction energy could be large !

A neutron star may have a much smaller total energy than the observed mass.

$\Downarrow$

A maximum energy of NS could be much smaller than its maximum mass.

Part II.
Conserved charges without symmetry

S. Aoki, T. Onogi and S. Yokoyama,
"Charge conservation, Entropy, and Gravitation",
arXiv:2010.07660[gr-qc].

Charge
$$\boxed{Q[v](t) = \int_{\Sigma_t} d^{d-1}x\,\sqrt{|g|}\,T^0{}_\nu v^\nu,}$$

$$\frac{dQ[v]}{dt} = \int_{\Sigma_t} d^{d-1}\vec{x}\sqrt{|g|}T^\mu{}_\nu \nabla_\mu v^\nu \neq 0$$
in generic spacetime w/o symmetry

sufficient condition for conservation
$$\boxed{T^\mu{}_\nu \nabla_\mu v^\nu = 0}$$
"conservation condition" for v

$$\Rightarrow \quad A_\mu{}^\nu \partial_\nu v^\mu + B_\mu v^\mu = 0 \qquad A_\mu{}^\nu := T_\mu{}^\nu,\ B_\mu = T^\alpha{}_\beta \Gamma^\beta_{\alpha\mu}$$

fix direction $v^\mu = v\delta_0^\mu$ $\Rightarrow$ $A^\mu \partial_\mu v + Bv = 0$ 1st order linear PDE

$$\Rightarrow \quad \frac{dx^\mu}{dt} = A^\mu(x), \qquad \frac{dv(t)}{dt} = -B(x)v(t)$$

initial value is given on a hyper surface at fixed t. $\Rightarrow$ A solution exists (at least locally in t).

simultaneous linear ODE

cf. Kodama vector is a solution for the spherically symmetric case.

$\frac{dQ[v]}{dt} = 0$ **we can define a generic conserved charge in general relativity.**

Hereafter, we consider the case where the vector is proportional to time direction, and write it as $v^\mu = \zeta^\mu$

$$S := Q[\zeta] = \int d^{d-1}x\, s^0 \qquad s^\mu = \sqrt{|g|} T^\mu{}_\nu \zeta^\nu.$$

What is this conserved charge ?

(1) This is not a Noether charge, since no symmetry exists.

(2) This is not an energy, but reduces to the energy for the Killing vector.

II-1. (a special type of) Perfect fluid

$$T^\mu{}_\nu = \rho n^\mu n_\nu + P\bar{g}^\mu{}_\nu$$

n^μ: unit time evolution vector $\quad n^0 = \dfrac{1}{N}$

$$ds^2 = -N^2(dx^0)^2 + \bar{g}_{ij}dx^i dx^j$$

$$\bar{g}^\mu{}_\nu := \delta^\mu_\nu + n^\mu n_\nu$$

conservation condition for $\zeta^\mu = -\beta n^\mu$

$$\rho n^\mu \partial_\mu \beta - \beta P K = 0 \qquad K := \bar{g}^\nu{}_\mu \nabla_\nu n^\mu = \nabla_\mu n^\mu$$

extrinsic curvature

charge density $s^0 := \rho\sqrt{-g}n^0\beta = u\beta$

$u := \rho v$: local energy

$v := \sqrt{\bar{g}}$: local volume

$\bar{g} := \det \bar{g}_{ij}$

Variations

$$\frac{ds^0}{d\eta} = \frac{du}{d\eta}\beta + u\frac{d\beta}{d\eta} = \left(\frac{du}{d\eta} + P\frac{dv}{d\eta}\right)\beta \qquad n^\mu = \frac{\partial x^\mu}{\partial \eta}$$

$$\because \quad \frac{d\beta}{d\eta} = \frac{\beta P K}{\rho} = \frac{\beta P}{u}\frac{dv}{d\eta} \qquad K = \frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}n^\mu) = \frac{d(\log\sqrt{\bar{g}})}{d\eta}$$

If we take $\beta = \dfrac{1}{T}$

$$\boxed{T\frac{ds^0}{d\eta} = \frac{du}{d\eta} + P\frac{dv}{d\eta}}$$

1st law of thermodynamics

Entropy $\quad S := Q[\zeta] = \int d^{d-1}x\, s^0$

entropy current density

$$s^\mu = \sqrt{|g|}T^\mu{}_\nu\zeta^\nu$$

conserved

conservation of Energy Momentum Tensor

$$(\nabla_\mu T^\mu{}_\nu) n^\nu = 0$$

$$n^\mu \partial_\mu \rho + (\rho + P) K = 0$$

$$\frac{d(\rho\beta)}{d\eta} + \rho\beta K = 0$$

$$\beta = \beta_0 \frac{\rho_0}{\rho} \exp\left[- \int_{\eta_0}^{\eta} d\eta K \right]$$

$$\frac{ds^0}{d\eta} = \frac{d(\rho\beta)}{d\eta} v + (\rho\beta) \frac{dv}{d\eta} = 0$$

entropy density is constant on the trajectory.

II-2. Homogeneous and Isotropic Universe

$$ds^2 = -dt^2 + a^2(t)\bar{g}_{ij}dx^i dx^j$$

Freedman-Lemaitre-Robertson-Walker metric

$$T^{\mu}{}_{\nu} = \rho n^{\mu} n_{\nu} + P g_{\mu\nu}$$

perfect fluid

$$\beta = \beta_0 \frac{\rho_0}{\rho} \exp\left[-\int_{\eta_0}^{\eta} d\eta K\right]$$

$$K = \frac{d(\log a^{d-1}(t))}{dt}$$

$$\rho(t)a^{d-1}(t)\beta(t) = \underline{u(t)\beta(t) = t\text{-independent}}$$

$$du \leq 0 \Rightarrow d\beta \geq 0$$

$$T ds^0 = du + P dv = 0$$

$$dv > 0 \text{ (expand)} \Rightarrow du \leq 0$$

T decreases as the Universe expands $P \neq 0$

(T: constant for P=0)

For the closed Universe, it expands, stops and contracts.

Entropy is conserved during this process.

II-3. Exact gravitational plane wave

$$ds^2 = e^{2\Omega(u)}(dx^2 - d\tau^2) + u^2(e^{2\beta(u)}dy^2 + e^{-2\beta(u)}dz^2) \qquad u = \tau - x$$

EMT $\quad T^x{}_x = T^\tau{}_x = -T^\tau{}_\tau = -T^x{}_\tau = \dfrac{e^{-2\Omega}}{4\pi u}\left(2\Omega' - u\beta'^2\right)$

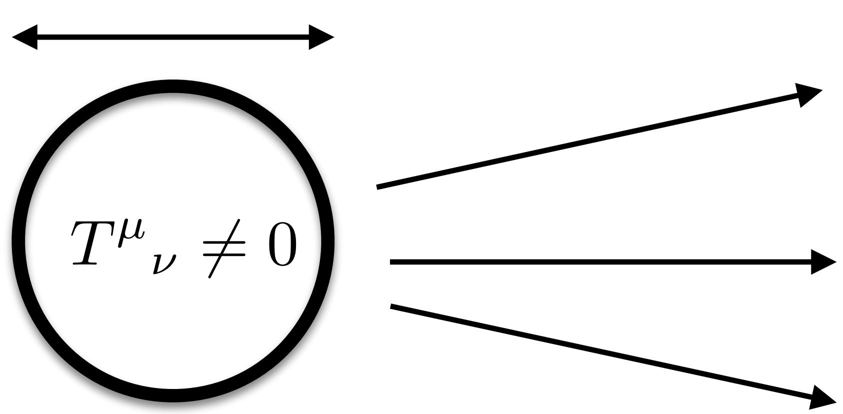
conservation vectors $\quad v_0^\mu = -\delta_\tau^\mu \qquad v_x^\mu = \delta_x^\mu$

conserved charges $\quad E = P_x = \dfrac{V_2}{4\pi}\displaystyle\int dx\, u\,(2\Omega' - u\beta'^2)$

$$2\Omega' - u\beta'^2 = 0$$
$$u\beta'' + 2\beta' - u^2\beta'^3 \neq 0$$

$\Rightarrow \quad E = P_x = 0 \qquad {}^\forall R_{ab} = 0, {}^\exists R_{abcd} \neq 0$

The Ricci flat gravitational wave does not carry energy/momentum.



$$T^\mu{}_\nu = 0, R_{\mu\nu\alpha\beta} \neq 0$$

II-4. Black hole entropy

Schwarzschild black hole

solution

$$T^{\mu}{}_0 = -\rho\delta^{\mu}_0, \quad \rho := \frac{(d-2)}{16\pi G_d}\frac{r_0^{d-3}\delta(r)}{r^{d-2}} \Rightarrow \zeta^{\mu} = -\zeta(\vec{x})\delta^{\mu}_0 \Rightarrow \zeta := \zeta(\vec{0})$$

$$\Rightarrow S = -\int d^{d-1}x\, T^0{}_0\zeta(\vec{x}) = E\zeta \qquad E := \frac{(d-2)V_{d-2}r_0^{d-3}}{16\pi G_d}$$

Assume $TdS = dE$

$$T := \frac{d-3-\frac{2\Lambda r_H^2}{d-2}}{4\pi r_H}$$

Hawking temperature

$$\frac{dE}{Tdr_H} = \frac{(d-2)V_{d-2}r_H^{d-3}}{4G}$$

$$S = E\zeta$$

$$u(r_H) = -1$$

outer horizon

$$\zeta + E\frac{d\zeta}{dE} = \frac{1}{T} \Rightarrow \zeta = \frac{1}{E}\int\frac{dE}{T} = \frac{1}{E}\int_0^{r_H}\frac{dE}{Tdr_H}dr_H = \frac{V_{d-2}r_H^{d-2}}{4G_dE}$$

$$\Rightarrow S = \frac{V_{d-2}r_H^{d-2}}{4G_d} = \frac{A_H}{4G_d}$$

Bekenstein-Hawking formula

However, entropy is localized at the origin, not at the horizon.

BTZ black hole

$$\zeta^\mu = -\zeta\delta^\mu_0 \qquad S = E\zeta$$

Assume

$$\left.\frac{\partial S}{\partial E}\right|_{P_\phi} = \frac{1}{T}$$

$$T = \frac{r_+^2 - r_-^2}{2\pi L^2 r_+}$$

Hawking temp.

$$r_\pm^2 = \frac{mL^2}{2}\left(1 \pm \sqrt{1-\left(\frac{J}{mL}\right)^2}\right)$$

horizon

$$E = \frac{r_+^2 + r_-^2}{8G_3L^2}, \quad P_\phi = \frac{2r_+r_-}{8G_3L}$$

$$S = \int_0^{r_+} dr_+ \left.\frac{dE}{Tdr_+}\right|_{P_\phi} = \frac{1}{4G_3}\int_0^{r_+}\frac{dr_+}{T}\left(r_+ + r_-\frac{dr_-}{dr_+}\right) = \frac{2\pi r_+}{4G_3} = \frac{A_H}{4G_3}$$

Bekenstein-Hawking

$$\frac{dP_\phi}{dr_+} = 0 \Rightarrow \frac{dr_-}{dr_+} = -\frac{r_-}{r_+}$$

Furthermore

$$\left.\frac{\partial S}{\partial P_\phi}\right|_E = -\frac{2\pi r_- L}{r_+^2 - r_-^2}$$

chemical potential

$$\mu_\phi := -T\left.\frac{\partial S}{\partial P_\phi}\right|_E = \frac{r_-}{r_+L}$$

$$TdS = dE - \mu_\phi dP_\phi$$

1st law of thermodynamics

Summary and Outlook

Part I.

covariant and universal definition of conserved energy for the 1st time

$$E = \int_{\Sigma(x^0)} d\Sigma_0 \sqrt{-g} T^0{}_\mu \xi^\mu$$

Killing

$$E_{\rm ours} = \frac{(d-2)V_{d-2} r_0^{d-3}}{16\pi G_d} \qquad E_{\rm Komar} = \frac{c(d-3)V_{d-2} r_0^{d-3}}{16\pi G_d}$$

	Neutral BH	Charged BH	BTZ BH	compact star
Ours	$E_{\rm ours}$	$+\infty$	$E = \frac{m}{8G_3}$ $P_\phi = \frac{J}{8G_3}$	$E_{\rm ours} - \Delta E$
Komar (Volume)	$\infty \ (\Lambda \neq 0)$ $E_{\rm Komar} \ (\Lambda = 0)$	$+\infty$	$E = \infty$ $P_\phi = \frac{cJ}{8G_3}$	$\infty \ (\Lambda \neq 0)$ $E_{\rm Komar} \ (\Lambda = 0)$
Komar (Surface)	$\infty \ (\Lambda \neq 0)$ $E_{\rm Komar} \ (\Lambda = 0)$	$+0$	$E = \infty$ $P_\phi = \frac{cJ}{8G_3}$	$\infty \ (\Lambda \neq 0)$ $E_{\rm Komar} \ (\Lambda = 0)$

failure of Stokes' theorem

Part II.

generic conserved charge = entropy $$S = \int_{\Sigma(x^0)} d\Sigma_0 \sqrt{-g}\, T^0{}_\mu \zeta^\mu$$

A correct understanding of general relativity after 105 years from Einstein.

Entropy is a source of the gravitational interaction.

Gravitational fields do not carry energy/entropy.

A total entropy in the whole system is always conserved,
as nothing can escape from a censorship of gravity.

天網恢恢疎にして漏らさず。

Gravity may provide a new tool to define entropy and temperature
in an arbitrary system.

Future investigations

Classical general relativity

binary stars How do they loose “energy” ?

gravitational collapse

Quantum gravity

Is it necessary to quantize gravity ?

Gravitational fields classically have no energy/entropy.

No exchange of energy/entropy between matters and gravitational fields.

If necessary, how can we quantize gravitons with no observed energy/entropy ?