Non-perturbative tests of duality cascades in three-dimensional supersymmetric gauge theories

Naotaka Kubo (YITP, Kyoto U.)

Based on: [Masazumi Honda-NK, 2010.15656]

YUKAWA INSTITUTE FOR THEORETICAL PHYSICS

(iTHEMS Math-Phys joint seminar, December 2020)

Part II

Non-perturbative tests

Part I

# of duality cascades in three-dimensional supersymmetric gauge theories

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Part I

Duality cascades in 3d supersymmetric gauge theories

Contents

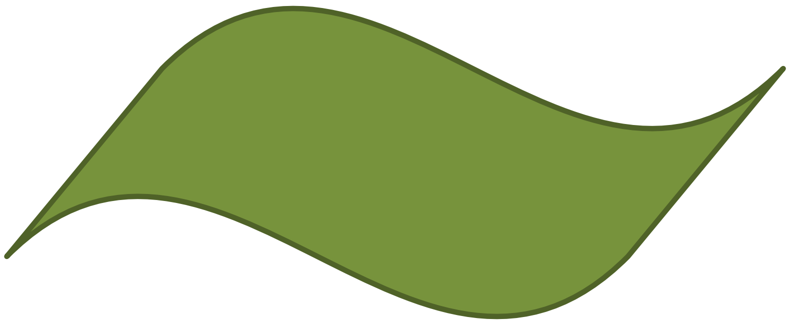
M2-brane

M-theory:

- A candidate of quantum gravity.
- Low-energy limit: supergravity on 11d.

Supergravity:

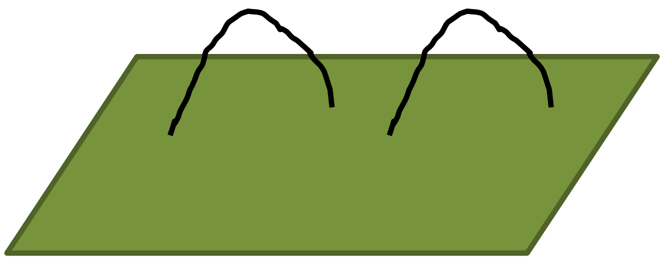
- 3d object called M2-brane



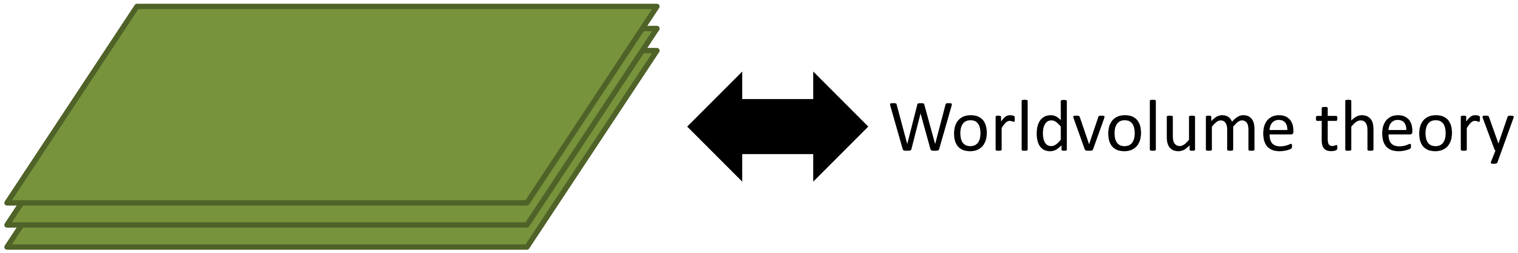
Worldvolume theory

The low-energy dynamics of brane can be captured by gauge theory = worldvolume theory.

Gauge theory is a specific class of quantum field theory (QFT).



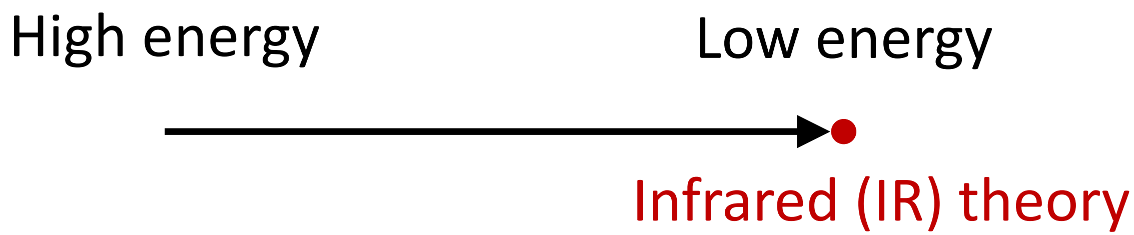
We consider the worldvolume theory of M2-branes.



Renormalization group (RG) flow

To specify a QFT, we need to fix the energy scale.
In other words, a QFT has energy scale as a parameter of it.

Renormalization group (RG) flow:



We want to know the IR theory of the worldvolume theory.

Supersymmetry (SUSY)

- The worldvolume theory has supersymmetry (SUSY).

High energy → Low energy

Worldvolume theory (SUSY) → IR theory (SUSY?)

- It was predicted that non-trivial mechanism called duality cascade occurs and sometimes SUSY is preserved.
- We predict that the duality cascade occurs for more general worldvolume theories.
- We performed non-perturbative tests by using the partition function.

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- Introduction
- Gauge theory, RG flow and supersymmetry
- Worldvolume theory and ABJ(M) theory
- Duality cascade
- Brane picture

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Gauge theory

In this talk, we consider gauge theory (as the worldvolume theory).

Data to define a gauge theory:

- Fiber bundle (with gauge group).
- Matter Content.
- A lagrangian.
- Energy scale.

In this talk:

- Gauge group plays an important role.
- We do not write matter content explicitly.
- We do not write a lagrangian explicitly.
- Energy scale plays an important role.

Supersymmetry

Supersymmetry:

Lagrangian is invariant under the exchange of bosons (scalar, vector) and fermions (spinor).

SUSY breaking:

Even if lagrangian has SUSY, sometimes the vacuum state is not SUSY.

Renormalization group (RG) flow

QFT depends on the energy level.

Space of quantum field theories:

High energy Low energy

Conformal field theory (CFT)

If we start with SUSY theory, there are two cases:

High energy Low energy

SUSY preserving:

SUSY SUSY+CFT=superconformal theory

High energy Low energy

SUSY breaking:

SUSY CFT

Contents

The worldvolume theory

The worldvolume theory of N M2-branes on $\mathbb{C}^4/\mathbb{Z}_k$:

- $\mathcal{N} = 3$ supersymmetry
- Gauge group: $\underline{\mathrm{U}(N) \times \mathrm{U}(N)}$
- Matter content: two bi-fundamental hypermultiplets
- Yang-Mills-Chern-Simons theory
- Chern-Simons level: k and $-k$

ABJM theory

This theory flows to a theory called ABJM theory.

[Aharony-Bergman-Jafferis-Maldacena 2008]

ABJM theory:

- $\mathcal{N} = 6$ superconformal
- Gauge group: $\mathrm{U}(N) \times \mathrm{U}(N)$
- Matter content: two bi-fundamental hypermultiplets
- Chern-Simons theory
- Chern-Simons level: k and $-k$

Worldvolume theory	$\longrightarrow$	ABJM theory
$\mathrm{U}(N) \times \mathrm{U}(N)$		$\mathrm{U}(N) \times \mathrm{U}(N)$
Chern-Simons level: $(k, -k)$		Chern-Simons level: $(k, -k)$

Rank deformation

We can add “fractional M2-branes”.

[Aharony-Bergman-Jafferis 2008]

The ranks of the worldvolume theory obtain one additional parameter:

$$\mathrm{U}(N) \times \mathrm{U}(N) \longrightarrow \mathrm{U}(N_1) \times \mathrm{U}(N_2)$$

Now, the worldvolume theory has three parameters:

- Chern-Simons level $(k, -k)$.
- Ranks (N_1, N_2).

SUSY breaking?

Now IR theory is non-trivial.

- When $|N_1 - N_2| \leq k$,

Worldvolume theory $\longrightarrow$ ABJ theory

$\mathrm{U}(N_1) \times \mathrm{U}(N_2)$ $\qquad$ $\mathrm{U}(N_1) \times \mathrm{U}(N_2)$

- However, when $|N_1 - N_2| > k$,

Worldvolume theory $\longrightarrow$ ABJ theory

$\mathrm{U}(N_1) \times \mathrm{U}(N_2)$ $\qquad$ $\mathrm{U}(N_1) \times \mathrm{U}(N_2)$

?

SUSY break?

What happens when $|N_1 - N_2| > k$?

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Duality cascade in 4d

In 4d $\mathcal{N} = 1$ SUSY $\mathrm{SU}(N+M) \times SU(N)$ gauge theory, non-trivial RG flow was found. [Klebanov-Strassler 2007]

$$\mathrm{SU}(N+M) \times SU(N)$$
$$\downarrow$$
$$\mathrm{SU}(N) \times SU(N-M)$$
$$\downarrow$$
$$\mathrm{SU}(N-M) \times SU(N-2M)$$
$$\vdots$$

Rule: $(N, M) \rightarrow (N-M, M)$

Duality cascade in 3d

It was conjectured that the similar story holds for the 3d worldvolume theory.

[Aharony-Hashimoto-Hirano-Ouyang 2009]
[Evslin-Kuperstein 2009]

$$\mathrm{U}(N) \times \mathrm{U}(N+M) \quad (M > k)$$

$$\downarrow$$

$$\mathrm{U}(N+k-M) \times \mathrm{U}(N)$$

$$\downarrow$$

$$\mathrm{U}(N+3k-2M) \times \mathrm{U}(N+k-M)$$

$$\vdots$$

Rule: $(N, M) \to (N+k-M, M-k)$

IR theory

There are two cases of endpoints for duality cascade.

$$\mathrm{U}(N) \times \mathrm{U}(N+M) \dashrightarrow \mathrm{U}(N') \times \mathrm{U}(N'+M')$$

$$\text{w/} \begin{cases} M' \leq k \\ N' + k - M' < 0 \end{cases}$$

- First case:

Worldvolume theory $\qquad\qquad$ ABJ theory

$$\mathrm{U}(N') \times \mathrm{U}(N'+M') \longrightarrow \mathrm{U}(N') \times \mathrm{U}(N'+M')$$

- Second case: SUSY breaking.

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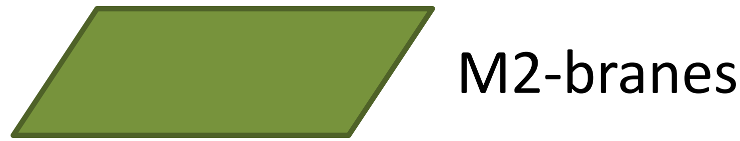
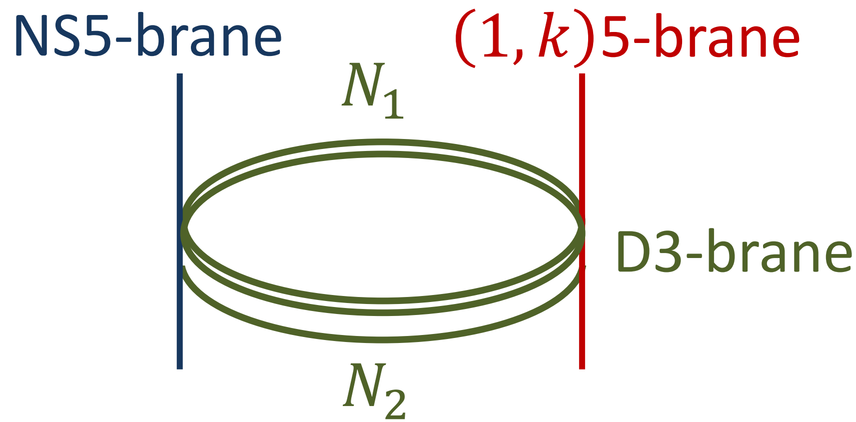
Type IIB string theory

The duality cascade and SUSY breaking can be captured in terms of brane construction in type-IIB string theory.

- M-theory



- Type-IIB string theory



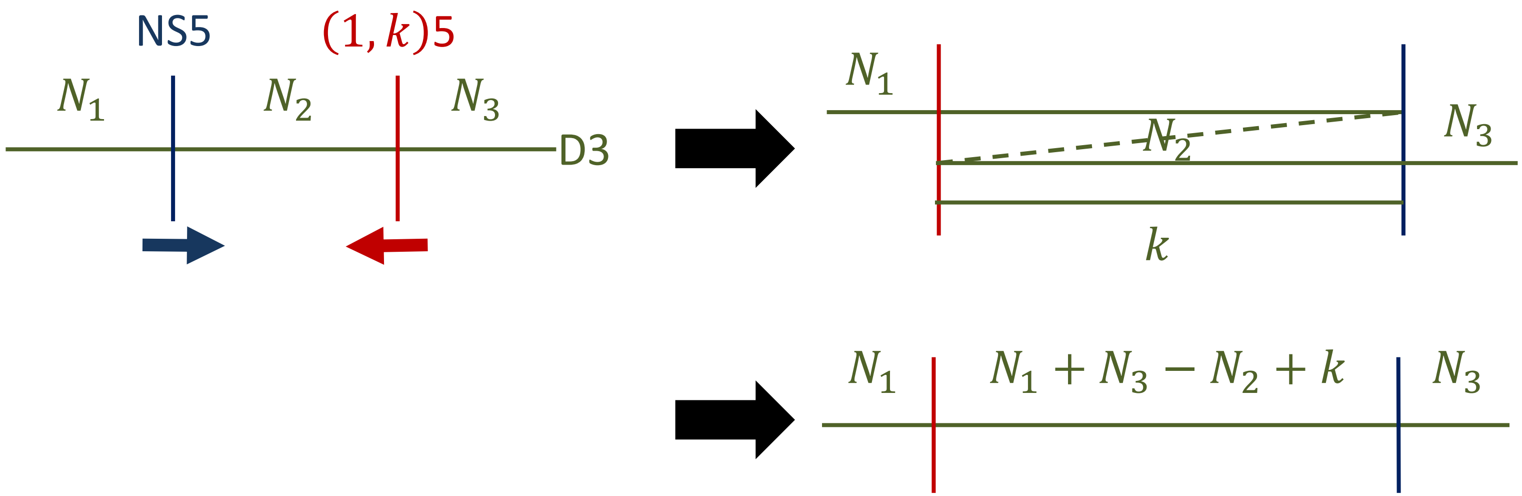
- Worldvolume theory

$\mathrm{U}(N_1) \times \mathrm{U}(N_2)$,
Chern-Simons level: $(k, -k)$

Hanany-Witten move

When an NS5-brane and a $(1,k)$5-brane pass through each other, the number of D3-branes stretched between them varies.

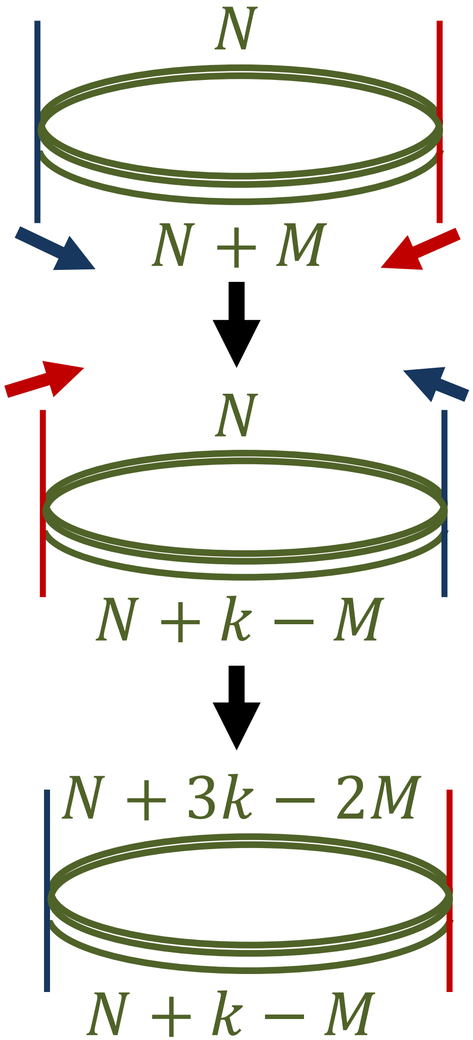
[Hanany-Witten 1997]



IR theory is invariant under continuous deformation of branes.

Duality cascade in brane picture

Duality cascade can be interpret as a sequence of Hanany-Witten moves.



$$\mathrm{U}(N) \times \mathrm{U}(N+M)$$

$$\downarrow$$

$$\mathrm{U}(N+k-M) \times \mathrm{U}(N)$$

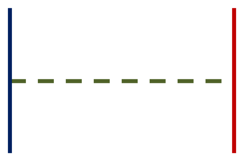
$$\downarrow$$

$$\mathrm{U}(N+3k-2M) \times \mathrm{U}(N+k-M)$$

$$\vdots$$

S-rule

When anti-D3-brane exist, a brane configuration no longer has SUSY.

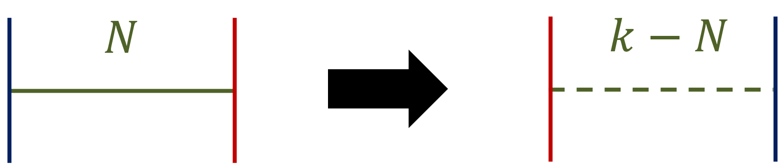


: SUSY break

The worldvolume theory also has no SUSY (s-rule).

[Bergman-Hanany-Karch-Kol 1999]

Combining the Hanany-Witten move, when $N > k$,

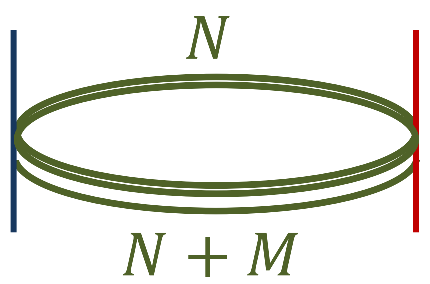


This implies that the SUSY breaking occurs for worldvolume theory on the left hand side.

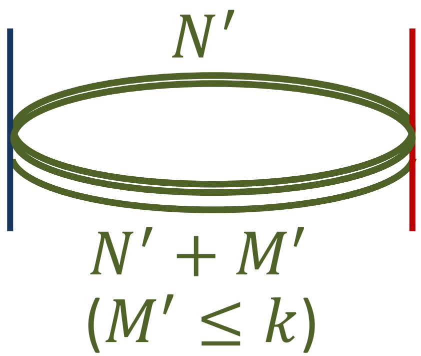
Prediction from brane picture

In summary,

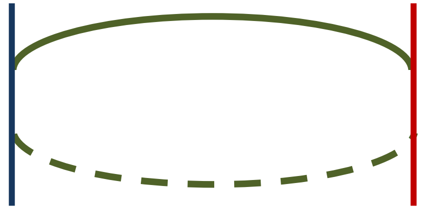
We start with a brane configuration:



After the sequence of Hanany-Witten transition,



or

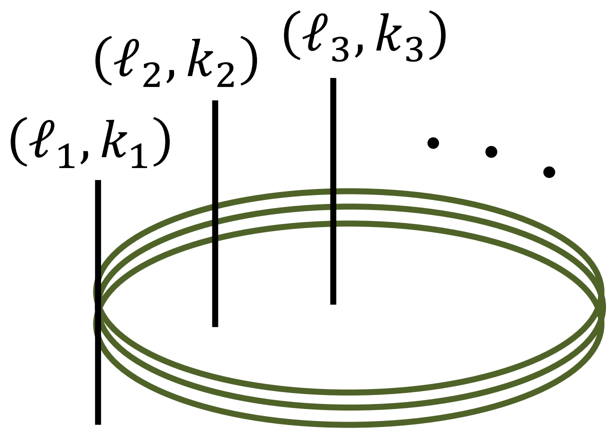


IR: ABJ theory with $\mathrm{U}(N') \times \mathrm{U}(N' + M')$

IR: SUSY break

General brane configuration

We can consider general types of 5-branes and increase the number of 5-branes.



The worldvolume theory is $\mathcal{N} = 4$ SUSY YM-CS theory with quiver gauge group.

General theory

If the behavior of the RG flow is captured in terms of brane picture, similar scenario should also holds for the general brane configurations.

- All the worldvolume theories which are related by the sequence of the Hanany-Witten transitions flow to the same IR theory.
- If anti-D3-branes appears, the SUSY is broken.

In our work, we perform non-perturbative test of this prediction by using the partition function.

Part II

Non-perturbative tests

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Partition function

$$\boldsymbol{\mathcal{T}} \coloneqq \text{a worldvolume theory on } S^3$$

The partition function on S^3 is defined using path integral:

$$Z[\boldsymbol{\mathcal{T}}] \coloneqq \int D\Phi \mathrm{e}^{-\mathrm{S}[\Phi]}$$

In a nutshell,

$$Z: \boldsymbol{\mathcal{T}} \to \mathbb{C}$$

Prediction from brane picture

The important properties of partition function:

- The absolute value of it does not depend on the energy scale.
- It becomes zero when SUSY breaking occurs.

Therefore, prediction for the partition function from brane picture is as follows.

$\widehat{\mathrm{HW}}[\mathcal{T}]$

$:= \{\text{Theories obtained by Hanany-Witten moves of } \mathcal{T}\}$

- $|Z[\mathcal{T}]| = |Z[\mathcal{T}']|$ for $\mathcal{T}' \in \widehat{\mathrm{HW}}[\mathcal{T}]$
- $Z[\mathcal{T}] = 0$ when $\widehat{\mathrm{HW}}[\mathcal{T}]$ includes anti-D3-branes

We showed that this equality exactly holds.

Localization technique

The partition function on S^3 reduces to matrix model by using so-called localization technique.

[Kapustin-Willett-Yaakov 2010]

Comment:
Localization technique≅Duistermaat–Heckman formula

Matrix model: Finite dimensional integral, well-defined.

Brane and matrix model

Brane configuration in type IIB string theory

Worldvolume theory

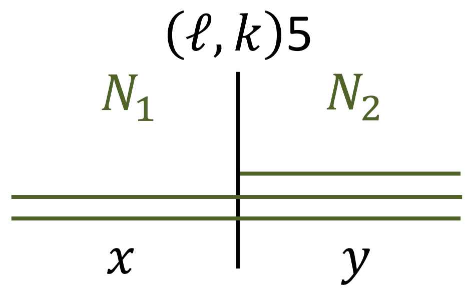
Partition function

Matrix model

Dictionally

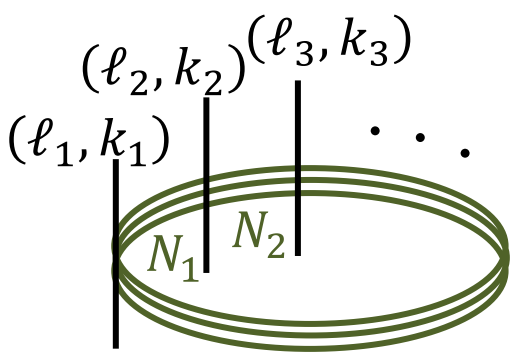
Dictionally from brane configurations to matrix models:

[Assel 2014]



$$Z_{(\ell,k)}\left(N_1,N_2;x,y\right)$$

$$Z_{(\ell,k)}\left(N_1,N_2;x,y\right)=\frac{1}{|\ell|^{\frac{N_1+N_2}{2}}}e^{\frac{i\pi k}{\ell}\left(\sum_m^{N_1}x_m^2-\sum_n^{N_2}y_n^2\right)}$$
$$\times\frac{\prod_{m<m'}^{N_1}2\sinh\frac{\pi(x_m-x_{m'})}{\ell}\prod_{n<n'}^{N_2}2\sinh\frac{\pi(y_n-y_{n'})}{\ell}}{\prod_m^{N_1}\prod_n^{N_2}2\cosh\frac{\pi(x_m-y_n)}{\ell}}$$



$$Z=\left(\prod_{j=1}^{n}\int\frac{d^{N_j}x^{(j)}}{N_j!}\right)\prod_{j=1}^{n}Z_{(\ell_j,k_j)}\left(N_j,N_{j+1};x^{(j)},x^{(j+1)}\right)$$

Contents

Simplify the problem

- $|Z[\boldsymbol{\mathcal{T}}]| = |Z[\boldsymbol{\mathcal{T}}']|$ for $\boldsymbol{\mathcal{T}}' \in \widehat{\mathrm{HW}}[\boldsymbol{\mathcal{T}}]$
- $Z[\boldsymbol{\mathcal{T}}] = 0$ when $\widehat{\mathrm{HW}}[\boldsymbol{\mathcal{T}}]$ includes anti-D3-branes

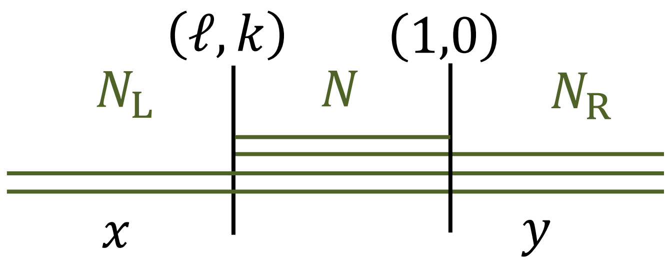
All theories in $\widehat{\mathrm{HW}}[\boldsymbol{\mathcal{T}}]$ is related by the sequence of the Hanany-Witten move by definition. Therefore, it is enough to show the following equality.

$$|Z[\boldsymbol{\mathcal{T}}]| = \begin{cases} |Z[\boldsymbol{\mathcal{T}}_{\mathrm{HW}}]| & (\boldsymbol{\mathcal{T}}_{\mathrm{HW}} \text{ does not include anti-D3-branes}) \\ 0 & (\boldsymbol{\mathcal{T}}_{\mathrm{HW}} \text{ include anti-D3-branes}) \end{cases}$$

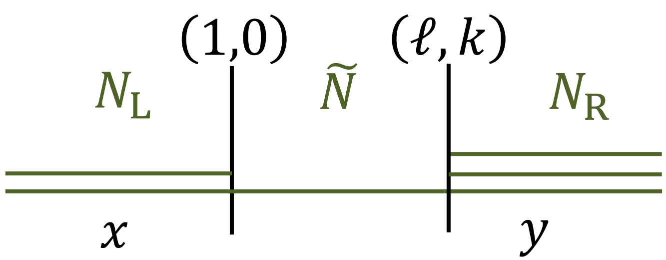
w/ $\boldsymbol{\mathcal{T}}_{\mathrm{HW}}$ is related to $\boldsymbol{\mathcal{T}}$ by a single Hanany-Witten move.

Simplify the problem: local theory

We can further reduce the problem to the local argument. Recall that there are local correspondence between brane configurations and matrix models. Therefore, let us focus on the local theory where Hanany-Witten move occurs.



$$Z(x,y)$$
$$= \int \frac{d^N\mu}{N!} Z_{(\ell,k)}\left(N_{\mathrm{L}}, N; x, \mu\right) Z_{(1,0)}\left(N, N_{\mathrm{R}}; \mu, y\right)$$



$(\widetilde{N} = N_L + N_R - N + k)$

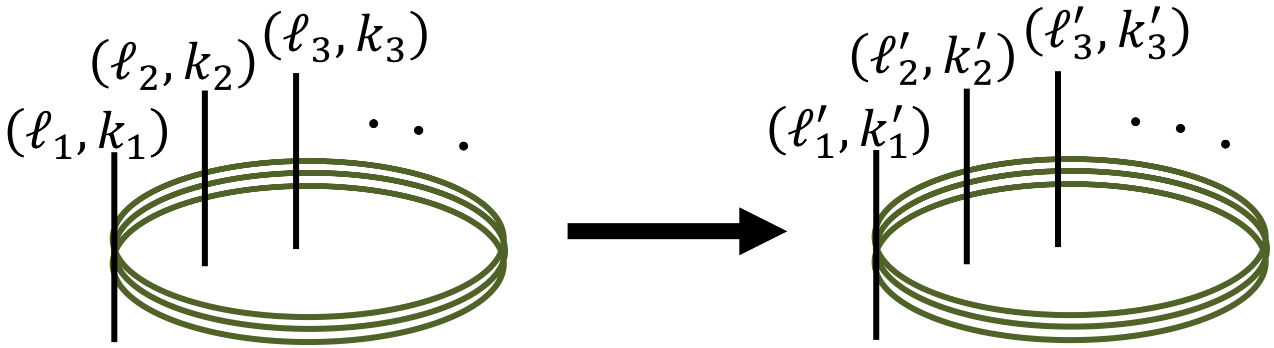
$$Z_{\mathrm{HW}}(x,y)$$
$$= \int \frac{d^{\tilde{N}}\nu}{\tilde{N}!} Z_{(1,0)}\left(N_{\mathrm{L}}, \tilde{N}; x, \nu\right) Z_{(\ell,k)}\left(\tilde{N}, N_{\mathrm{R}}; \nu, y\right)$$

Therefore, it is sufficient to show that

$$\boxed{|Z(x,y)| = \begin{cases} |Z_{\mathrm{HW}}(x,y)| & \left(\tilde{N} \geq 0\right) \\ 0 & \left(\tilde{N} < 0\right) \end{cases}}$$

SL(2, $\mathbb{Z}$) duality

SL(2, $\mathbb{Z}$) duality:



$$\begin{pmatrix} \ell'_i \\ k'_i \end{pmatrix} = A \begin{pmatrix} \ell_i \\ k_i \end{pmatrix}, \qquad A \in \mathrm{SL}(2, \mathbb{Z})$$

The matrix models corresponding to dual configurations are equal.

[Assel 2014]

Therefore, we can fix the 5-branes without loss of generality.

Computation

We prove the identity:

$$\boxed{|Z(x,y)| = \begin{cases} |Z_{\mathrm{HW}}(x,y)| & \left(\tilde{N} \geq 0\right) \\ 0 & \left(\tilde{N} < 0\right) \end{cases}}$$

$(\widetilde{N} = N_L + N_R - N + k)$

where

$$Z(x,y) = \int \frac{d^N \mu}{N!} Z_{(\ell,k)}\left(N_{\mathrm{L}}, N; x, \mu\right) Z_{(1,0)}\left(N, N_{\mathrm{R}}; \mu, y\right)$$

$$Z_{\mathrm{HW}}(x,y) = \int \frac{d^{\tilde{N}} \nu}{\tilde{N}!} Z_{(1,0)}\left(N_{\mathrm{L}}, \tilde{N}; x, \nu\right) Z_{(\ell,k)}\left(\tilde{N}, N_{\mathrm{R}}; \nu, y\right)$$

$$Z_{(\ell,k)}\left(N_1, N_2; x, y\right) = \frac{1}{|\ell|^{\frac{N_1+N_2}{2}}} e^{\frac{i\pi k}{\ell}\left(\sum_m^{N_1} x_m^2 - \sum_n^{N_2} y_n^2\right)}$$

$$\times \frac{\prod_{m<m'}^{N_1} 2\sinh\frac{\pi(x_m - x_{m'})}{\ell} \prod_{n<n'}^{N_2} 2\sinh\frac{\pi(y_n - y_{n'})}{\ell}}{\prod_m^{N_1}\prod_n^{N_2} 2\cosh\frac{\pi(x_m - y_n)}{\ell}}$$

Fermi gas formalism

Important technique: Fermi gas formalism

We rewrite the integrand in terms of quantum mechanics.

[Marino-Putrov 2011]
[NK 2020]

$$[\hat{q}, \hat{p}] = i\hbar,\ \hbar = \frac{\ell}{2\pi k}$$

Combination of Cauchy determinant formula
and Vandermonde determinant formula

$$\frac{\prod_{m<m'}^{N_1} 2\sinh\frac{\pi(x_m - x_{m'})}{\ell} \prod_{n<n'}^{N_2} 2\sinh\frac{\pi(y_n - y_{n'})}{\ell}}{\prod_m^{N_1}\prod_n^{N_2} 2\cosh\frac{\pi(x_m - y_n)}{\ell}} = (-1)^{N_1(N_2-N_1)} \det \begin{pmatrix} \left[\frac{e^{\frac{\pi M}{\ell}(x_m - y_n)}}{2\cosh\frac{\pi(x_m-y_n)}{\ell}}\right]_{m,n}^{N_1\times N_2} \\ \left[e^{\frac{2\pi}{\ell}t_{M,j}y_n}\right]_{j,n}^{(N_2-N_1)\times N_2} \end{pmatrix}$$

$$= \det \begin{pmatrix} \left[\hbar\langle x_n|\frac{1}{2\cosh\frac{\hat{p}-i\pi M}{2}}|y_m\rangle\right]_{n,m}^{N\times(N+M)} \\ \left[\frac{\hbar}{\sqrt{k}}\langle\langle 2\pi i t_{M,j}|y_m\rangle\right]_{j,m}^{M\times(N+M)} \end{pmatrix}$$

w/ $t_{M,j} = \frac{M+1}{2} - j$

Similarity transformation

Fresnel factor can be used to obtain delta function:

$$e^{-\frac{i}{2\hbar}\hat{p}^2}e^{-\frac{i}{2\hbar}\hat{q}^2}f\left(\hat{p}\right)e^{\frac{i}{2\hbar}\hat{q}^2}e^{\frac{i}{2\hbar}\hat{p}^2} = f\left(\hat{q}\right)$$

$$e^{-\frac{i}{2\hbar}\hat{p}^2}e^{-\frac{i}{2\hbar}\hat{q}^2}|p\rangle\rangle = \frac{1}{\sqrt{i}}e^{\frac{i}{2\hbar}p^2}|p\rangle$$

Now we obtain delta functions coming from inner products of position eigenvectors. We can perform integration by using the delta functions.

Residue computation

$$\int d^N\mu \, \frac{\prod_{m<m'}^{N_\mathrm{L}} 2\sinh\frac{\pi(\alpha_m-\alpha_{m'})}{\ell} \prod_{n<n'}^{N} 2\sinh\frac{\pi(\mu_n-\mu_{n'})}{\ell}}{\prod_m^{N_\mathrm{L}}\prod_n^N 2\cosh\frac{\pi(\alpha_m-\mu_n)}{\ell}}$$

$$\times\left(\prod_n^{N_\mathrm{R}}\langle\mu_n|\frac{1}{2\cosh\left(\frac{1}{2\hbar}\hat{q}+\frac{i\pi M}{2}\right)}|\beta_n\rangle\right)\left(\prod_j^{N-N_\mathrm{R}}\langle\mu_{N_\mathrm{R}+j}|\underbrace{-\frac{\ell}{k}it_{M,j}}_{\text{Imaginary}}\rangle\right)$$

We should be careful when we use delta functions.

$$\begin{aligned}\int_\mathbb{R} dx f(x)\langle x|a+ib\rangle &= \frac{1}{2\pi}\int_\mathbb{R} dpdx f(x)\, e^{ip(x-a-ib)}\\ &= \frac{1}{2\pi}\left(\int_\mathbb{R} dpdx f(x+ib)\, e^{ip(x-a)} + 2\pi i \,\mathrm{sgn}(b)\sum_j R_j\int_\mathbb{R} dp e^{ip(z_j-a-ib)}\right)\\ &= f(a+ib) + 2\pi i\,\mathrm{sgn}(b)\sum_j R_j\langle z_j|a+ib\rangle\end{aligned}$$

R_j: Residue of $f(x)$ at $0\le \mathrm{Im}(x)\le b$

Residue computation

Poles:

$$\prod_{n}^{N_{\mathrm{L}}} \frac{1}{2\cosh \frac{\pi\left(\alpha_n - \mu_{N_{\mathrm{R}}+j}\right)}{\ell}}$$

Residue:

$$\mu_{N_{\mathrm{R}}+k+j} = \alpha_n + \left(m - \frac{1}{2}\right) i\ell, \quad m \in \mathbb{Z}$$

$(0 \le j \le N - N_{\mathrm{R}} - k)$

We should choose different α_n for different μ_{N_R+k+j}

Therefore, if $N_{\mathrm{L}} < N - N_{\mathrm{R}} - k$, matrix model vanishes.

$$\boxed{\widetilde{N} < 0}$$

Summary

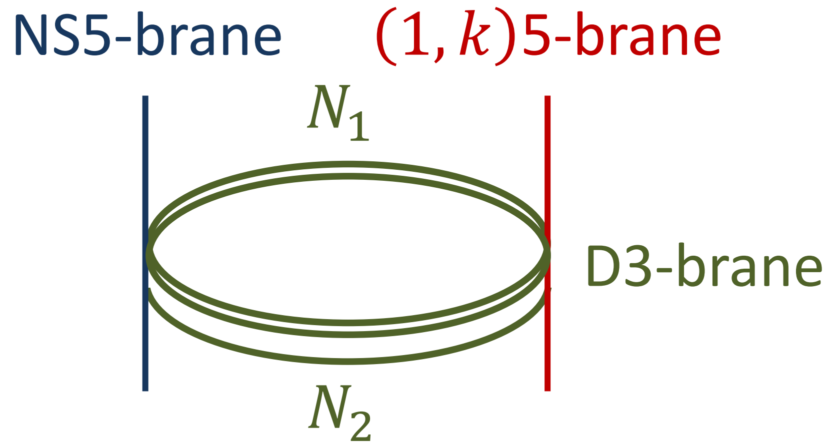
- We have studied the conjectural duality cascade in the 3d $\mathcal{N} = 3$ SUSY gauge theories.
- The brane dynamics seem to capture the behavior of the RG flow of the gauge theory.
- We performed non-perturbative tests. The result is completely consistent with the study from the viewpoint of brane picture.

Future work

- Our computation is for local theory. Therefore, perhaps our computation technique can be applied for more general theories.
- Arguments in brane picture suggests that the duality cascade occurs for the worldvolume theories of more general brane configurations. Non-perturbative test?
- In my work, we focused on the IR structure of gauge theories. It is important to study the RG flow directly.

Back up

Brane configuration



	0-2	3	4	5	6	7	8	9
D3	○				○			
NS5	○	$/_{37}$	$/_{48}$	$/_{59}$		$/_{37}$	$/_{48}$	$/_{59}$
(1,k)5	○	○	○	○				

Quiver diagram

ABJM theory: