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Quantum kinetic theory for chiral and spin transport in relativistic heavy ion collisions and core-collapse supernovae

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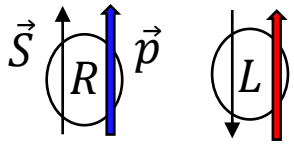
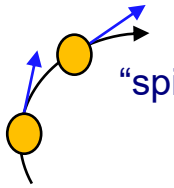
Quantum transport for massless fermions

- Weyl fermions :
  chirality=helicity
  (no chirality mixing)



$$\mathbf{J}_V = \mathbf{J}_R + \mathbf{J}_L$$
$$\mathbf{J}_5 = \mathbf{J}_R - \mathbf{J}_L$$

(spin current)



"spin enslavement"

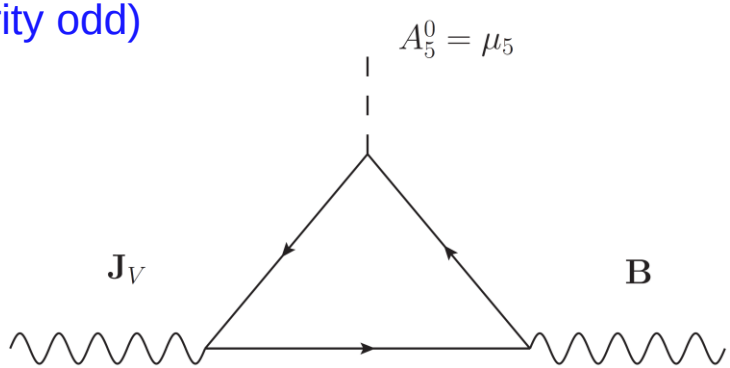
- Chiral anomaly : $\partial_\mu J^\mu_{R/L} = \pm \dfrac{\mathbf{E}\cdot\mathbf{B}}{4\pi^2}$ ⟹ $\partial_\mu J^\mu_5 = \dfrac{\mathbf{E}\cdot\mathbf{B}}{2\pi^2}$

S. Adler, J. Bell, R. Jackiw, 69
K. Fujikawa, 79

- Classical transport : $\mathbf{J}_V = \sigma_e \mathbf{E}$
  (Ohmic current)

- Anomalous transport (in chiral matter):

Chiral magnetic effect (CME) : $\mathbf{J}_V = \dfrac{\boxed{\mu_5}}{2\pi^2}\mathbf{B}$ (parity odd)



A. Vilenkin, PRD 20, 1807 (1979). PRD 22, 3080 (1980)
K. Fukushima, D. Kharzeev, H. Warringa, PRD78, 074033 (2008)
D. E. Kharzeev, L. D. McLerran, H. J. Warringa, Nucl.Phys.A 803 (2008) 227-253

Chiral magnetic/vortical effect (CME/CVE)

- Anomalous transport of Weyl fermions :

A. Vilenkin, PRD 20, 1807 (1979). PRD 22, 3080 (1980)
K. Fukushima, D. Kharzeev, H. Warringa, PRD78, 074033 (2008)
D. T. Son, P. Surowka, PRL 103, 191601 (2009)
K. Landsteiner, E. Megias, F. Pena-Benitez, PRL107,021601(2011)

$$J^\mu_{R/L} = \pm\frac{\mu_{R/L}}{4\pi^2}B^\mu \pm \left(\frac{T^2}{12} + \frac{\mu^2_{R/L}}{4\pi^2}\right)\omega^\mu$$

$\mu_{V/5} = (\mu_R \pm \mu_L)/2$

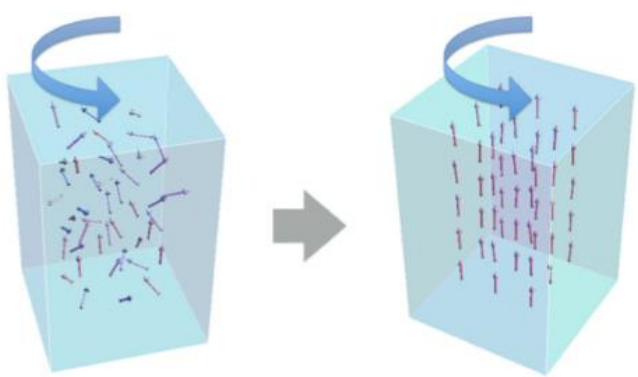
CME : $\mathbf{J}_V = \frac{1}{2\pi^2}\mu_5\mathbf{B}$

CVE : $\mathbf{J}_V = \frac{1}{\pi^2}\mu_5\mu_V\,\boldsymbol{\omega}$

CSE : $\mathbf{J}_5 = \frac{1}{2\pi^2}\mu_V\mathbf{B}$

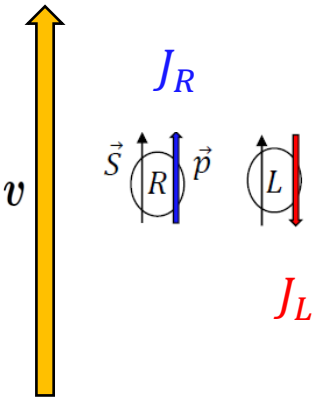
aCVE : $\mathbf{J}_5 = \left(\frac{\mu_V^2+\mu_5^2}{2\pi^2} + \frac{T^2}{6}\right)\boldsymbol{\omega}$

- An “intuitive” picture for CVE : Barnett effect + spin enslavement



vorticity :

$$\boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\nabla}\times\boldsymbol{v}$$

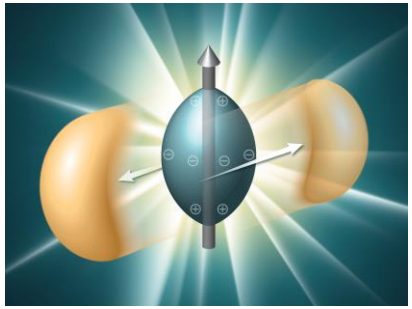


M. Matsuo et al, fphy.2015.00054

Where to find anomalous transport in the real world?

- Nuclear physics : relativistic heavy ion collisions
  (strong B field & vorticity generated)

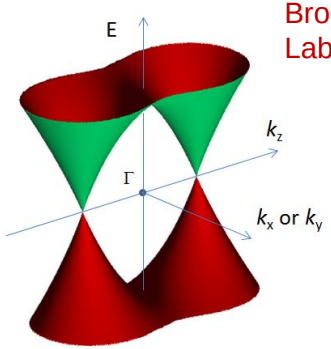
  e.g. D. Kharzeev, et.al, Prog.Part.Nucl.Phys. 88, 1 (2016)



Brookhaven National Laboratory

- Condensed matter : Weyl semimetals

  e.g. Q. Li, et.al., Nature Phys. 12 (2016) 550-554



Borisenko et al., Phys. Rev. Lett. 113, 027603 (2014)

- Astrophysics : core-collapse supernovae

  e.g. N. Yamamoto, Phys. Rev. D93, 065017 (2016)

credit : RIKEN

Outline

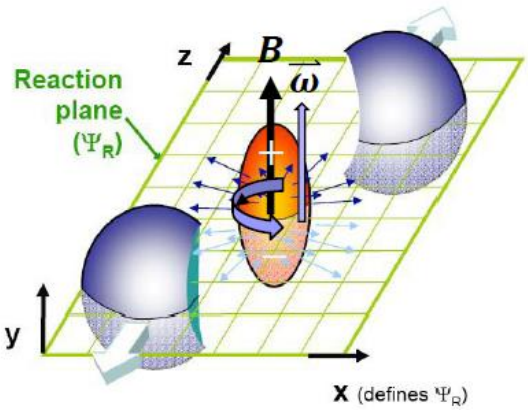
- Chiral and spin transport in relativistic heavy ion collisions (HIC)
  - CME & spin polarization of hadrons
  - Dynamical spin polarization : non-equilibrium spin transport
  - Quantum kinetic theory (QKT)
  - Spin-1/2 massless fermions
  - Extensions (massive fermions, spin-1 massless bosons)

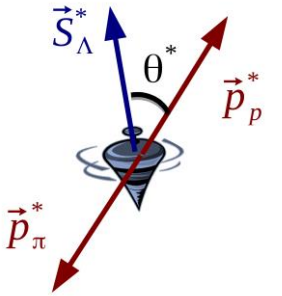
- Neutrino transport in core-collapse supernovae (CCSN)
  - Chiral transport in CCSN
  - Chiral magneto-hydrodynamics (CMHD) & inverse energy cascade
  - Chiral radiation hydrodynamics (CRHD)

- Conclusions & outlook

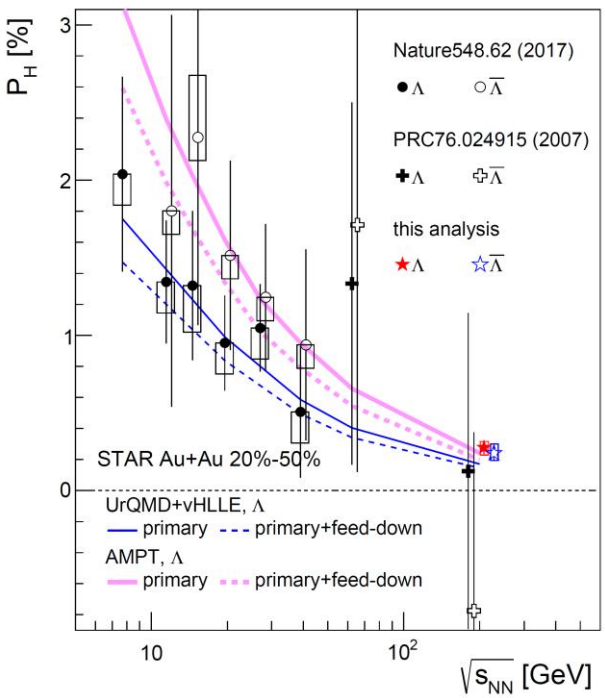
Spin polarization in heavy ion collisions

- Strong magnetic/vortical fields in HIC
- CME signal is still inconclusive





$\Lambda \to p + \pi^-$



- Global polarization of Λ hyperons :

❖ Statistical model/Wigner-function approach (in equilibrium):

$$\mathcal{P}^\mu(\mathbf{q}) = \frac{\int d\Sigma \cdot q J_5^\mu(q,X)}{2m \int d\Sigma \cdot \mathcal{N}(q,X)} \approx \frac{1}{8m}\epsilon^{\sigma\mu\nu\rho} q_\sigma \frac{\int d\Sigma \cdot q \omega_{\nu\rho} f_q^{(0)}(1-f_q^{(0)})}{\int d\Sigma \cdot q f_q^{(0)}},$$

$$\boxed{\omega_{\nu\rho} = \frac{1}{2}\big(\partial_\rho(u_\nu/T) - \partial_\nu(u_\rho/T)\big)}.$$

thermal vorticity

F. Becattini, et al., Ann. Phys. 338, 32 (2013)

R. Fang, et al., PRC 94, 024904 (2016)

Relativistic angular momentum

- Relativistic angular momentum for fermions : $\mathcal{L} = \bar{\psi}\left(\frac{i\hbar}{2}\gamma^\mu \overleftrightarrow{\partial}_\mu - m\right)\psi,$

  (review : E. Leader & C. Lorce, 13)

  $\overleftrightarrow{\partial}_\mu = \overrightarrow{\partial}_\mu - \overleftarrow{\partial}_\mu.$

- Noether theorem :

$$\delta\psi \to \frac{\tilde{\omega}_{\mu\nu}}{2}\left(\hat{J}^{\mu\nu} - i\Sigma^{\mu\nu}\right)\psi, \quad \Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu], \quad \hat{J}^{\mu\nu} = x^\mu\partial^\nu - x^\nu\partial^\mu. \quad \Rightarrow \quad \partial_\lambda M_C^{\lambda\mu\nu} = 0.$$

- Canonical AM tensor :

$$M_C^{\lambda\mu\nu} = M_S^{\lambda\mu\nu} + M_O^{\lambda\mu\nu},$$

$$M_S^{\lambda\mu\nu} = \frac{1}{2}\bar{\psi}\{\gamma^\lambda, \Sigma^{\mu\nu}\}\psi = \boxed{-\frac{1}{2}\epsilon^{\lambda\mu\nu\rho}\bar{\psi}\gamma_\rho\gamma_5\psi},$$ related to the axial-charge current

$$M_O^{\lambda\mu\nu} = \frac{i}{2}\bar{\psi}\gamma^\lambda\left(x^\mu\overleftrightarrow{\partial}^\nu - x^\nu\overleftrightarrow{\partial}^\mu\right)\psi = x^\mu T_C^{\lambda\nu} - x^\nu T_C^{\lambda\mu},$$

- Canonical EM tensor : $T_C^{\mu\nu} = T^{\mu\nu} + T_A^{\mu\nu}, \quad T^{\mu\nu} = \frac{i\hbar}{4}\bar{\psi}\gamma^{\{\mu}\overleftrightarrow{\partial}^{\nu\}}\psi, \quad T_A^{\mu\nu} = \frac{i\hbar}{4}\bar{\psi}\gamma^{[\mu}\overleftrightarrow{\partial}^{\nu]}\psi.$

- EM & AM conservation :

$$\partial_\mu T_C^{\mu\nu} = 0, \quad \partial_\lambda M_C^{\lambda\mu\nu} = 0. \quad \Rightarrow \quad \underbrace{\boxed{-\frac{\hbar}{2}\epsilon^{\lambda\mu\nu\rho}\partial_\lambda J_{5\rho}}}_{\text{spin}} + \underbrace{\boxed{2T_A^{\mu\nu}}}_{\text{orbit}} = 0$$

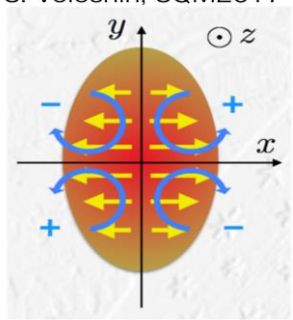
Local (longitudinal) polarization : a sign problem

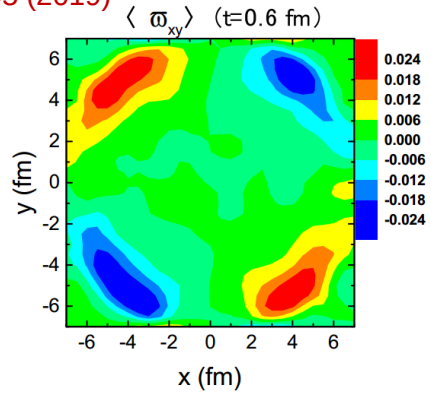
- Local vorticity away from central rapidity :

  transverse expansion :
  longitudinal vorticity

D.-X. Wei, W.-T. Deng, X.-G. Huang, PRC 99, 014905 (2019)

S. Voloshin, SQM2017

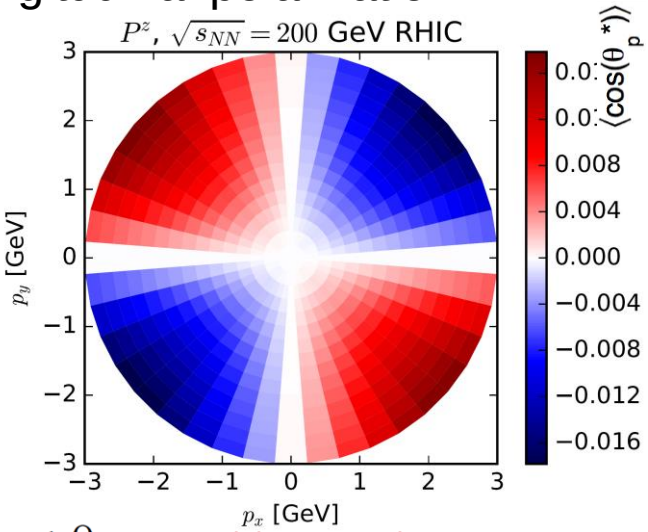




- Sign problem for longitudinal polarization :

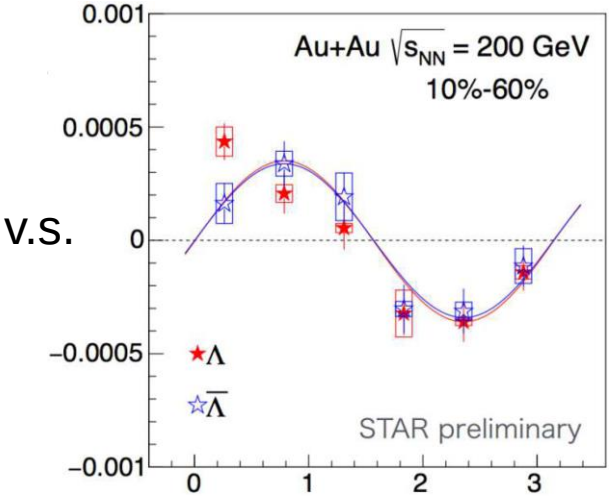
Spin harmonics :

$$\frac{dP^z}{2\pi d\phi} = f_{z,0} + 2 f_{z,2} \sin(2\phi)$$



$f_{z,2}^{\mathrm{th}} < 0$ F. Becattini, I. Karpenko, PRL 120, 012302 (2018)

V.S.



$f_{z,2}^{\mathrm{exp}} > 0$

(same structure, opposite signs!)

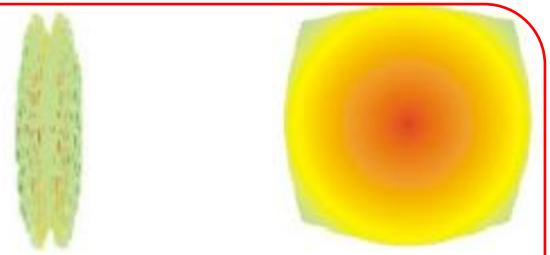
Dynamical evolution of the spin

- Disagreements also appear for transverse polarization
- (local) polarization may not be solely contributed by thermal vorticity

  ⇒ non-equilibrium effects may play a role

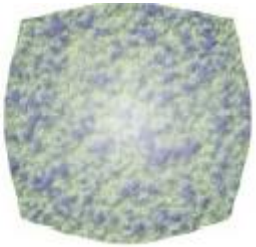
  W. Florkowski, et al., PRC100, 054907 (2019)
  H.-Z. Wu, et al., PRR 1, 033058 (2019)
- Current theoretical studies :

Initial states

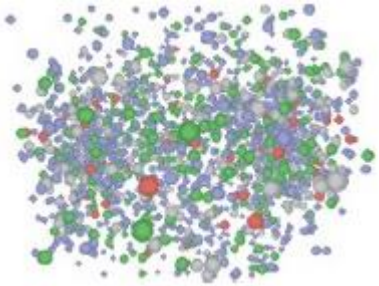


pre-equilibrium phase/thermaliation

QGP



hadronization



hadronic gas/freeze-out

Initial polarization :
Hard scattering with
$b \neq 0$
Z.-T. Liang, X.-N. Wang,
PRL 94, 102301 (2005)

Dyanamical spin polarization
in between?

Polarization of hadrons
in equilibrium :
e.g. statistical model

F. Becattini, et al.,
Ann. Phys. 338, 32 (2013)

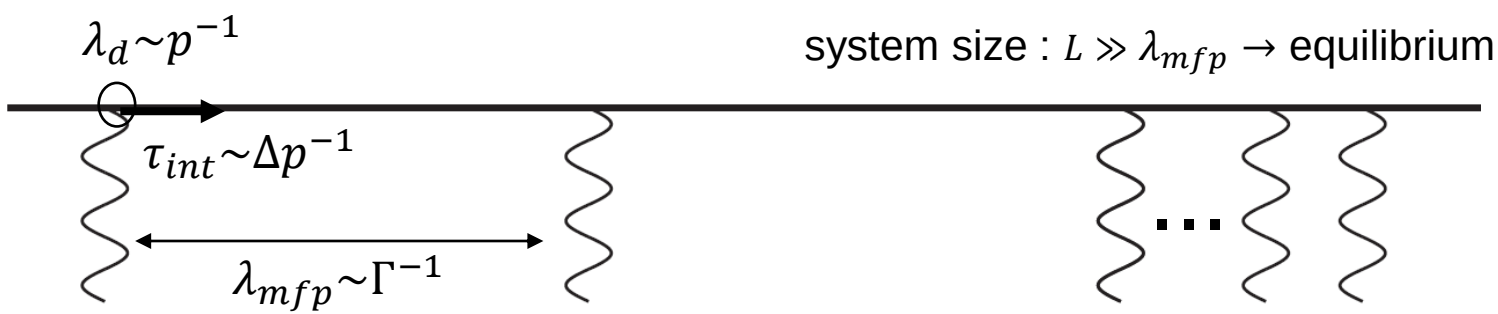
Transport theories

- Kinetic theory : microscopic theory for quasi-particles in phase space

❖ Boltzmann (Vlasov) Eq. : $q^\mu \Delta_\mu f(q,X) = q^\mu \mathcal{C}_\mu[f], \quad \Delta_\mu = \partial_\mu + F_{\nu\mu}\frac{\partial}{\partial q^\nu}.$

❖ Physical quantities : $J^\mu(X) = \int \frac{d^3\boldsymbol{q}}{(2\pi)^3}\frac{q^\mu}{E_q} f(q,X), \quad T^{\mu\nu}(X) = \int \frac{d^3\boldsymbol{q}}{(2\pi)^3}\frac{q^\mu q^\nu}{E_q} f(q,X).$

❖ Validity : mean free path ≫ de Broglie wavelength & mean free time ≫ int. time



- Conservation laws : $\partial_\mu J^\mu = 0, \quad \partial_\mu T^{\mu\nu} = F^{\nu\rho} J_\rho.$

❖ Near equilibrium : kinetic theory ⇒ hydrodynamics

Could we apply kinetic theory to HIC?

- Although QGP is strongly coupled, we may still learn some useful physics from the weakly coupled approach based on QCD.

  (e.g. thermalization or hydrodynamization of HIC with kinetic theory)

  P. B. Arnold, G. D. Moore, L. G. Yaffe, JHEP 0301(2003) 030. A. Kurkela & Y. Zhu, PRL. 115, 182301 (2015)

- Primary goal : to construct a framework for tracking the spin transport of a probe fermion (s quark) traversing a weakly coupled medium (wQGP).

- Relativistic kinetic theory :
  pros : microscopic theory, non-equilibrium, phase-space evolution
  cons : weakly coupled, weak background fields

- Quantum kinetic theory (QKT) : charge (energy-density) + spin dof

  Cooper-Frye formula for spin :

$$\mathcal{P}^{\mu}(\mathbf{q}) = \frac{\int d\Sigma \cdot q \boxed{J_5^{\mu}(q,X)}}{2m \int d\Sigma \cdot \mathcal{N}(q,X)}$$

  F. Becattini, et al., Ann. Phys. 338, 32 (2013)
  R. Fang, et al., PRC 94, 024904 (2016)

- Other approaches : microscopic models, spin hydro, holography etc.

  e.g. J.-j. Zhang, et al., PRC 100, 064904 (2019)
  W. Florkowski, et al., PRC97, 041901 (2018)
  K. Hattori, et al., PLB 795, 100 (2019)
  A.D. Gallegos, U. Gürsoy, JHEP 11 (2020) 151

Chiral kinetic theory (CKT)

- To study anomalous transport from kinetic theory :

❖ Standard kinetic theory : $q^\mu \Delta_\mu f = q^\mu \mathcal{C}_\mu, \quad \Delta_\mu = \partial_\mu + F_{\nu\mu}\frac{\partial}{\partial q^\nu}$ ⟹ $\partial_\mu J^\mu = 0$

❖ CKT : ? ⟹ $\partial_\mu J^\mu = \frac{\hbar}{4\pi^2}\mathbf{E}\cdot\mathbf{B}$ (for right-handed fermions)

- Liouville's theorem : $\frac{d}{dt}f(x,\mathbf{p}) = (\partial_t + \dot{\mathbf{x}}\cdot\nabla_{\mathbf{x}} + \dot{\mathbf{p}}\cdot\nabla_{\mathbf{p}})\, f = 0$
- Quantum correction : classical action + Berry phase

D. T. Son & N. Yamamoto, PRL. 109, 181602 (2012)

M. Stephanov & Y. Yin, PRL. 109, 162001 (2012)

❖ EOM :

$$\begin{aligned}\dot{\mathbf{x}} &= \hat{\mathbf{p}} + \hbar\dot{\mathbf{p}}\times\mathbf{\Omega}_{\mathbf{p}},\\ \dot{\mathbf{p}} &= \mathbf{E} + \dot{\mathbf{x}}\times\mathbf{B}.\end{aligned}$$

Berry curvature : $\mathbf{\Omega}_{\boldsymbol{p}} = \frac{\lambda_{\mathrm{f}}\boldsymbol{p}}{|\boldsymbol{p}|^3}, \quad \lambda_{\mathrm{f}} = \pm\frac{1}{2}.$

❖ CKT (collisionless) :

$$\left[(1+\hbar\mathbf{B}\cdot\mathbf{\Omega}_{\mathbf{p}})\,\partial_t + (\tilde{\mathbf{v}} + \hbar\mathbf{E}\times\mathbf{\Omega}_{\mathbf{p}} + \hbar(\tilde{\mathbf{v}}\cdot\mathbf{\Omega}_{\mathbf{p}})\mathbf{B})\cdot\boldsymbol{\nabla} + \left(\tilde{\mathbf{E}} + \tilde{\mathbf{v}}\times\mathbf{B} + \hbar(\tilde{\mathbf{E}}\cdot\mathbf{B})\mathbf{\Omega}_{\mathbf{p}}\right)\cdot\frac{\partial}{\partial\mathbf{p}}\right] f = 0$$

effective velocity/E field : $\tilde{\mathbf{v}} = \partial\epsilon_{\mathbf{p}}/\partial\mathbf{p}$, $\tilde{\mathbf{E}} = \mathbf{E} - \partial\epsilon_{\mathbf{p}}/\partial\mathbf{x}$

$\epsilon_{\mathbf{p}} = |\mathbf{p}| - \boxed{\hbar\frac{\mathbf{B}\cdot\mathbf{p}}{2|\mathbf{p}|^3}}$ magnetic-moment coupling

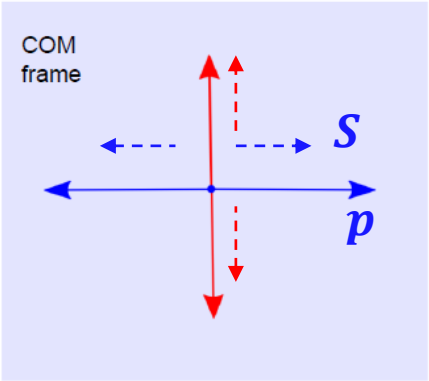
❖ Field-theory derivation : Wigner functions + hbar expansion

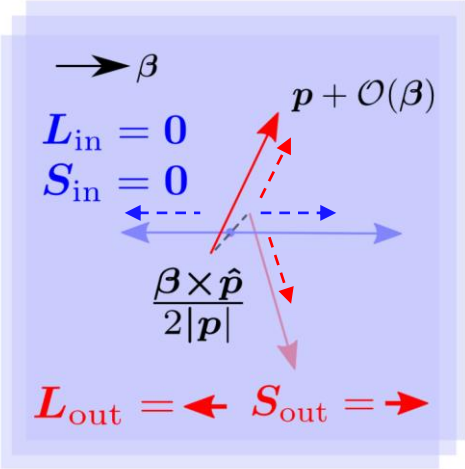
J.-W. Chen, et al., PRL. 110, 262301 (2013) D. T. Son & N. Yamamoto, PRD. 87, 085016 (2013)

Y. Hidaka, S. Pu, DY, PRD 95, 091901 (2017), PRD 97, 016004 (2018)

Side-jump phenomena

- Side jumps : spin enslavement + angular momentum (AM) conservation





❖ frame-dep of $f(p, x)$: modified Lorentz transformation.

J.-Y. Chen, et al., PRL. 113, 182302 (2014)
J.-Y. Chen, D. T. Son, and M. A. Stephanov, PRL. 115, 021601 (2015).

❖ quantum corrections on the collision term : self-energy gradients

Y. Hidaka, S. Pu, DY, PRD 95, 091901 (2017)

❖ In general, side jumps can also occur for free massless particles with spin

Wigner functions (WFs)

- lesser (greater) propagators :

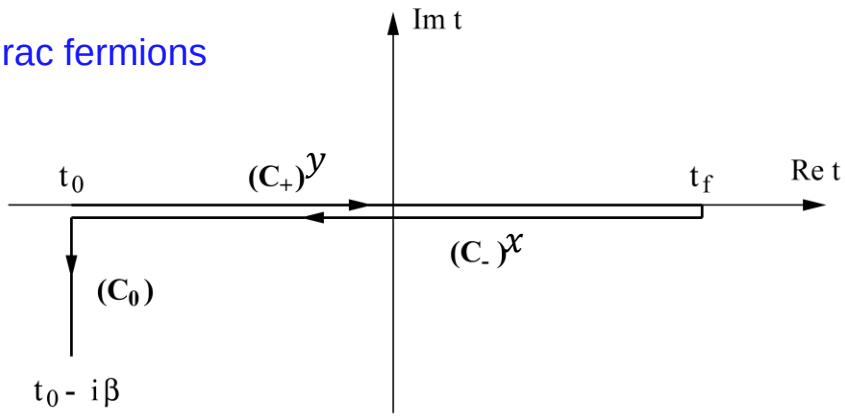
Dirac fermions

$$\tilde{S}^{>}_{\alpha\beta}(x,y) = \langle \psi_\alpha(x) U^\dagger(x,y) \bar{\psi}_\beta(y) \rangle$$

$$\tilde{S}^{<}_{\alpha\beta}(x,y) = \langle \bar{\psi}_\beta(y) \boxed{U(y,x)} \psi_\alpha(x) \rangle$$

gauge link

$$X = \frac{x+y}{2}, \; Y = x - y$$



review : J. Blaizot, E. Iancu, Phys.Rept. 359 (2002) 355-528

Wigner functions : $S^{<(>)}_{\alpha\beta}(q,X) = \int d^4Y e^{\frac{iq\cdot Y}{\hbar}} \tilde{S}^{<(>)}_{\alpha\beta}(x,y)$

- hbar expansion : $X \gg Y \to q_\mu \gg \partial_\mu$ (weak fields)
- Kadanoff-Baym (KB) equations :

$$(\not{\Pi} - m) S^< + \gamma^\mu i \frac{\hbar}{2} \nabla_\mu S^< = \frac{i\hbar}{2} \boxed{\left(\Sigma^< \star S^> - \Sigma^> \star S^< \right)}$$

(systematic include collisions)

$$\nabla_\mu = \Delta_\mu + \mathcal{O}(\hbar^2),$$
$$\Delta_\mu = \partial_\mu + F_{\nu\mu} \partial/\partial q_\nu$$
$$\Pi^\mu = q^\mu + \mathcal{O}(\hbar^2)$$
$$A \star B = AB + \frac{i\hbar}{2} \{A, B\}_{\text{P.B.}} + \mathcal{O}(\hbar^2)$$

Quantum corrections for WFs

- R-handed WF up to $\mathcal{O}(\hbar)$: $S^{<}(q,X) = \bar{\sigma}^{\mu}\grave{S}_{\mu}(q,X)$,

$$\grave{S}^{<\mu}(q,X) = 2\pi \mathrm{sgn}(q\cdot n)\left(q^{\mu}\delta(q^2)f_q^{(n)} + \hbar\delta(q^2)S^{\mu\nu}_{(n)}\mathcal{D}_{\nu}f_q^{(n)} + \hbar\epsilon^{\mu\nu\alpha\beta}q_{\nu}F_{\alpha\beta}\frac{\partial\delta(q^2)}{2\partial q^2}f_q^{(n)}\right),$$

$$\mathcal{D}_{\beta}f_q^{(n)} = \Delta_{\beta}f_q^{(n)} - \Sigma^{<}_{\beta}(1-f_q^{(n)}) + \Sigma^{>}_{\beta}f_q^{(n)}.$$

side-jump term :
magnetization current
CVE or anomalous hall effects

modification on
the on-shell condition
CME in equilibrium

spin tensor : $S^{\mu\nu}_{(n)} = \dfrac{\epsilon^{\mu\nu\alpha\beta}}{2(q\cdot n)}q_{\alpha}n_{\beta}$ J.-Y. Chen, et al., PRL. 113, 182302 (2014)

Y. Hidaka, S. Pu, DY,
PRD 95, 091901 (2017),
PRD 97, 016004 (2018)

- Frame vector n^{μ} : freedom for decomposition & a local observer we choose

- The full WF has to be frame independent ⇒ $f_q^{(n)}$ is frame dependent
- The modified frame transformation :

$$f_q^{(n')} = f_q^{(n)} + \frac{\hbar\epsilon^{\nu\mu\alpha\beta}q_{\alpha}n'_{\beta}n_{\mu}}{2(q\cdot n)(q\cdot n')}\mathcal{D}_{\nu}f_q^{(n)}$$

⇒ $f^{(n')}(q,X) = f^{(n)}(q+\hbar\delta q, X+\hbar\delta X)$

(side jumps)

CKT with collisions

- CKT with collisions $(\partial_\rho n^\mu = 0)$:

Y. Hidaka, S. Pu, DY, PRD 95, 091901 (2017), PRD 97, 016004 (2018)

$$\boxed{\delta\left(q^2 - \hbar\frac{B\cdot q}{q\cdot n}\right)}\left\{\left[q\cdot\Delta + \hbar\frac{S^{\mu\nu}_{(n)}E_\mu}{(q\cdot n)}\Delta_\nu + \hbar S^{\mu\nu}_{(n)}(\partial_\mu F_{\rho\nu})\partial^\rho_q\right]f^{(n)}_q - \tilde{\mathcal{C}}\right\} = 0,$$

magnetic-moment coupling

($F^{\mu\nu} = 0$: the quantum corrections only appear in collisions)

- Quantum corrections on the collision term :

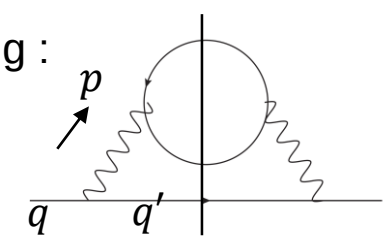
$$\tilde{\mathcal{C}} = q\cdot\mathcal{C} + \hbar\frac{S^{\mu\nu}_{(n)}E_\mu}{(q\cdot n)}\mathcal{C}_\nu + \hbar S^{\alpha\beta}_{(n)}\boxed{\left((1-f^{(n)}_q)\Delta_\alpha\Sigma^<_\beta - f^{(n)}_q\Delta_\alpha\Sigma^>_\beta\right)},$$

induced by inhomogeneity of the medium

$$\mathcal{C}_\beta = \boxed{\Sigma^<_\beta}(1-f^{(n)}_q) - \boxed{\Sigma^>_\beta}f^{(n)}_q.$$

also include hbar corrections

2-2 scattering :



- Solve $f_{R/L}$ and put them back to WFs ⟹ $\mathcal{A}_\mu(q,X) = (\grave{S}^<_{\mu\mathrm{R}} - \grave{S}^<_{\mu\mathrm{L}})/2$

Anomalous (chiral) hydro from CKT

- Anomalous hydrodynamics (R-handed) : $\partial_\mu T^{\mu\nu} = F^{\nu\rho} J_\rho, \quad \partial_\mu J^\mu = \frac{\hbar}{4\pi^2}(\mathbf{E}\cdot\mathbf{B})$
- Constitutive equations:

$$T^{\mu\nu} = \underset{\mathcal{O}(1)}{u^\mu u^\nu \epsilon - pP^{\mu\nu}} + \underset{\mathcal{O}(\hbar)}{\Pi^{\mu\nu}_{\rm non}} + \underset{\mathcal{O}(1)+\mathcal{O}(\hbar)}{\Pi^{\mu\nu}_{\rm dis}}, \quad J^\mu = \underset{\mathcal{O}(1)}{N_0 u^\mu} + \underset{\mathcal{O}(\hbar)}{v^\mu_{\rm non}} + \underset{\mathcal{O}(1)+\mathcal{O}(\hbar)}{v^\mu_{\rm dis}}, \quad P^{\mu\nu} = \eta^{\mu\nu} - u^\mu u^\nu.$$

e.g. D. T. Son and P. Surowka, PRL 103, 191601 (2009)

depending on interactions

- Transport coefficients from equilibrium WFs : Y. Hidaka, S. Pu, DY, PRD 97, 016004 (2018)

$$T^{\mu\nu} = \int \frac{d^4q}{(2\pi)^4}\left[q^\mu \grave{S}^{<\nu} + q^\nu \grave{S}^{<\mu}\right], \quad J^\mu = 2\int \frac{d^4q}{(2\pi)^4}\grave{S}^{<\mu}.$$

$$\Rightarrow \quad v^\mu_{\rm non} = \hbar\sigma_B B^\mu + \hbar\sigma_\omega \omega^\mu \qquad \Pi^{\mu\nu}_{\rm non} = \hbar\xi_\omega\left(\omega^\mu u^\nu + \omega^\nu u^\mu\right) + \hbar\xi_B\left(B^\mu u^\nu + B^\nu u^\mu\right)$$

$$\sigma_\omega = \frac{T^2}{12}\left(1 + \frac{3\bar{\mu}^2}{\pi^2}\right), \quad \sigma_B = \frac{\mu}{4\pi^2}, \quad \xi_\omega = \frac{T^3}{6}\left(\bar{\mu} + \frac{\bar{\mu}^3}{\pi^2}\right), \quad \xi_B = \frac{T^2}{24}\left(1 + \frac{3\bar{\mu}^2}{\pi^2}\right).$$

CVE CME

(agree with those from different approaches, e.g. Son & Surowka, 09. K. Landsteiner, et.al. Lect. Notes, 13.)

Entropy current and AM conservation

- Entropy-density current in equilibrium :

Boltzmann H function : $\mathcal{H}(f) = -f\ln f - (1-f)\ln(1-f)$

from WF : $s^{\mu}_{\mathrm{Rleq}} = \frac{1}{T}\left(u^{\mu}p_R + T^{\mu\nu}_{R\mathrm{leq}}u_{\nu} - \mu_R J^{\mu}_{R\mathrm{leq}} + \hbar D_{BR}B^{\mu} + \hbar D_{\omega R}\omega^{\mu}\right),$

$$D_{BR} = \frac{1}{8\pi^2}\left(\mu_R^2 + \frac{\pi^2T^2}{3}\right) = \frac{\xi_{BR}}{T}, \quad D_{\omega R} = \frac{1}{12}\left(T^2\mu_R + \frac{\mu_R^3}{\pi^2}\right) = \frac{\xi_{\omega R}}{2T}.$$

$\Rightarrow$ $\partial_{\mu}s^{\mu}_{R\mathrm{leq}} = 0$

no entropy production

DY, PRD98, 076019 (2018)

See also D. T. Son and P. Surowka, PRL 103, 191601 (2009)

- Anti-sym EM tensor : $T^{\mu\nu}_{A} = \frac{\hbar}{2}N_A\left(\omega^{\mu}u^{\nu} - \omega^{\nu}u^{\mu}\right)$ from side-jumps

DY, PRD98, 076019 (2018)

$$M^{\lambda\mu\nu}_{\mathrm{spin}}(X) = \frac{\hbar}{2}\epsilon^{\lambda\mu\nu\rho}\left(N_A u_{\rho} + \hbar\sigma_{BA}B_{\rho} + \hbar\sigma_{\omega A}\omega_{\rho}\right)$$

CSE & CVE

$\Rightarrow$ spin $-\frac{\hbar}{2}\epsilon^{\lambda\mu\nu\rho}\partial_{\lambda}J_{5\rho}$ + orbit $2T^{\mu\nu}_{A} = 0$

- $\mathcal{O}(\hbar)$: spin-orbit cancellation
- Higher orders : we need higher-order WFs.

Massive fermions

- Anomalous transport for massive fermions :
  - Chiral symmetry is explicitly broken
  - Chirality imbalance is washed out by mass in equilibrium. ⟹ CME/CVE vanish
  - CSE & ACVE still exist : $\boldsymbol{J}_5 = \sigma_{B5}\boldsymbol{B} + \sigma_{\omega 5}\boldsymbol{\omega} \xrightarrow[m \to \infty]{} \boldsymbol{J}_5 \to 0$

E. Gorbar, et al., PRD88, 025025 (2013)
M. Buzzegoli, E. Grossi, F. Becattini, JHEP 10, 091 (2017)
S. Lin and L. Yang, PRD98, 114022 (2018)

- QKT for massive fermions (e.g. for strange quarks)?
  - Chirality mixing : work in the vector/axial-vector bases.
  - Spin is no longer enslaved by chirality : a new dynamical dof
  - To track both vector/axial charges and spin polarization
  - Underlying quantum effects : chiral anomaly, spin-orbit int., etc.
  - Axial kinetic theory (AKT) : a scalar + an axial-vector equations

K. Hattori, Y. Hidaka, DY, PRD100, 096011 (2019)
DY, K. Hattori, and Y. Hidaka, JHEP 20, 070 (2020)

(a modified relativistic BMT eq.)

V. Bargmann, L. Michel, and V.L. Telegdi, PRL 2, 435 (1959).

$$\boxed{\Box^{(n)}\mathcal{A}^\mu} = \boxed{\hat{\mathcal{C}}^\mu_{\rm cl}} + \boxed{\hbar\hat{\mathcal{C}}^{(n)\mu}_{\rm Q}}$$

spin diffusion spin polarization

N. Weickgenannt, et al., PRD 100 (2019), 056018.
J.-H. Gao & Z.-T. Liang, , PRD100 (2019), 056021.
Z. Wang, X. Guo, S. Shi, and P. Zhuang, PRD 100, 014015 (2019)
S. Li and H.-U. Yee, PRD100, 056022 (2019))
N. Weickgenannt, et al., arXiv:2005.01506
Z. Wang, X. Guo, P. Zhuang, arXiv:2009.10930

(see also my talk at Spin & Hydrodynamics online workshop by ect* on YouTube)

Anomalous transport of photons

- Photonic CVE : $\boxed{j^{\pm}_{\rm CVE}} = \pm\frac{T^2}{6}\omega$

N. Yamamoto, PRD96, 051902 (2017)
See also : A. Avkhadiev and A. V. Sadofyev, PRD96, 045015 (2017)
X.-G. Huang and A. V. Sadofyev, JHEP 03, 084 (2019)

Chern-Simons (CS) currents for R/L-handed photons

$$j^{+\mu}_{\rm CVE} - j^{-\mu}_{\rm CVE} \sim \mathcal{K}^\mu \equiv A_\nu \tilde{F}^{\mu\nu}$$

- The gauge-inv. conserved quantity for Maxwell's equation :

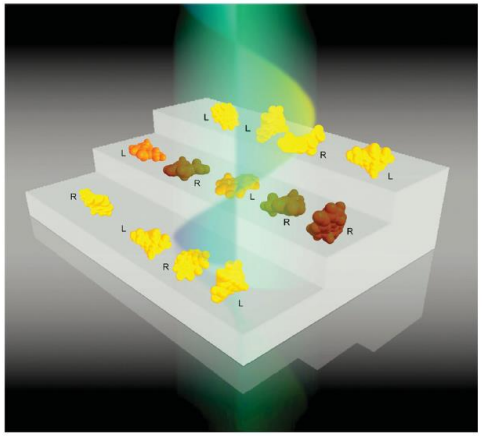
zilch : $\partial_\mu Z^{\mu\nu\rho} = 0, \quad Z_{\mu\nu\rho} \equiv \frac{1}{2}\left[F_\mu{}^\alpha \partial_\rho \tilde{F}_{\nu\alpha} - (\partial_\rho F_\nu{}^\alpha)\tilde{F}_{\mu\alpha}\right].$

D. Lipkin, J. Math. Phys. 5, 696 (1964)
T. Kibble, J. Math. Phys. 6, 1022 (1965)

- Optical chirality : $Z_{000} = \frac{1}{2}(\mathbf{B}\cdot\dot{\mathbf{E}} - \mathbf{E}\cdot\dot{\mathbf{B}})$

Y. Tang & A. Cohen, PRL. 104, 163901 (2010).

(asymmetry in the rates of excitation of chiral molecules)



- Zilch vortical effect (ZVE) :

$$Z^i \equiv Z^{i00} = \frac{8\pi^2 T^4 \omega^i}{45}$$

(zilch current)

M. Chernodub, A. Cortijo, K. Landsteiner, PRD98, 065016 (2018)
C. Copetti, J. Fernandez-Pendas, PRD 98, 105008 (2018) (from Kubo formula)

WFs & QKT for photons

- The full lesser propagator for photons (Coulomb gauge) : K. Hattori, Y. Hidaka, N. Yamamoto, DY, arXiv:2010.13368, JHEP XX (2021)

$$G^{<}_{\mu\nu} \equiv G^{\mathrm{R}<}_{\mu\nu} + G^{\mathrm{L}<}_{\mu\nu}$$

$$= 2\pi\delta(q^2)\mathrm{sgn}(q\cdot n)\left[\left(\boxed{P^{(n)}_{\mu\nu}} f_{\mathrm{V}} - \frac{\hbar q_{\perp(\mu} S^{(n)}_{\nu)\alpha}\partial^{\alpha}}{2(q\cdot n)^2} f_{\mathrm{A}}\right) - \mathrm{i}\underbrace{\left(\boxed{S^{(n)}_{\mu\nu}} f_{\mathrm{A}} + \frac{\hbar q_{\perp[\mu}\partial_{\perp\nu]}}{2(q\cdot n)^2} f_{\mathrm{V}}\right)}_{\text{for CS \& Zilch currents}}\right],$$

P-even

P-odd

$$f_{\mathrm{A}} \equiv (f_{\mathrm{R}} - f_{\mathrm{L}})/2 \text{ and } f_{\mathrm{V}} \equiv (f_{\mathrm{R}} + f_{\mathrm{L}})/2$$

➢ L.O. polarization tensors : $P^{(n)}_{\mu\nu} = n_\mu n_\nu - \eta_{\mu\nu} - \hat{q}_{\perp\mu}\hat{q}_{\perp\nu}$, $\boxed{S^{(n)}_{\mu\nu} = \frac{\epsilon_{\mu\nu\alpha\beta} q^{\alpha} n_{\beta}}{q\cdot n}}$.

spin tensor for photons

- CVE & ZVE : $\mathcal{K}^{\mu}_{\mathrm{eq}} = \frac{2}{9} T^2 \omega^{\mu}$ & $Z^{\alpha}_{\mathrm{eq}}(X) = \frac{8\pi^2}{45\hbar^2} T^4 \omega^{\alpha}$ see also X.-G. Huang, et al., JHEP 10, 117 (2020)

- Effective QKT for photons (gluons):

$$\left.\begin{aligned} q\cdot\partial f^{(\gamma)}_{\mathrm{A}} &= \mathcal{C}^{(\gamma)}_{\mathrm{A}} \\ &+ \\ \Box^{(n)}\mathcal{A}^{\mu} &= \hat{\mathcal{C}}^{\mu}_{\mathrm{cl}} + \hbar\hat{\mathcal{C}}^{(n)\mu}_{\mathrm{Q}} \end{aligned}\right\}$$

Coupled spin evolution in wQGP

Applications in astrophysics

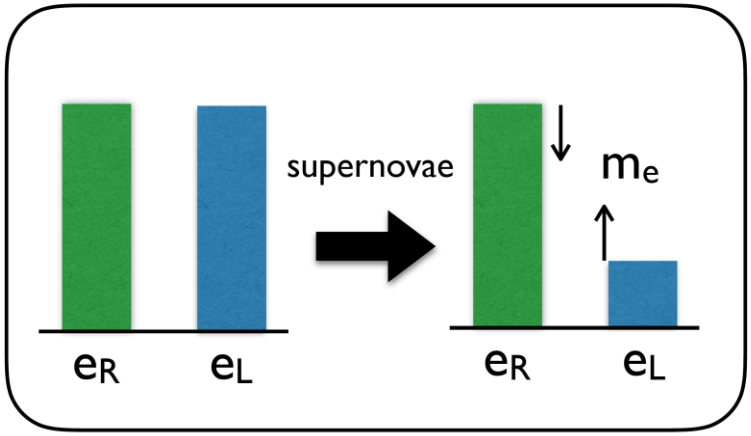
- Core-collapse supernovae (CCSN) could be ideal for studying the quantum transport of chiral matter. N. Yamamoto, PRD93, 065017 (2016)

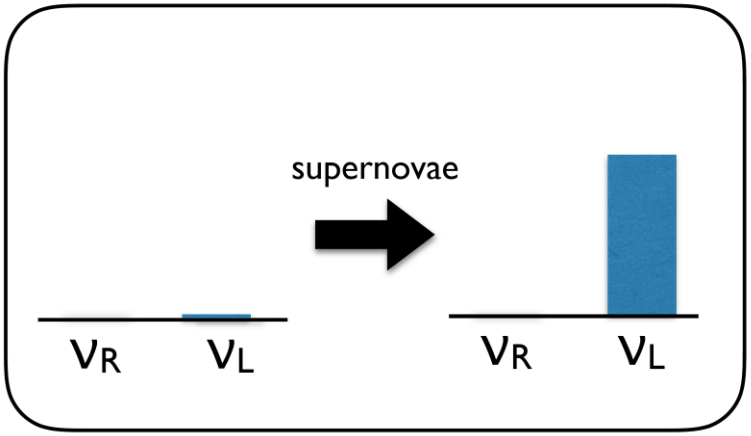
  Y. Masada, K. Kotake, T. Takiwaki, and N. Yamamoto, PRD98, 083018 (2018)

- Intrinsic chiral matter created in nature : weak interaction
- Neutrino/electron transport in supernovae :

an innate lefthander

$$p + e^{\mathrm{L}} \to n + \nu_e^{\mathrm{L}}$$

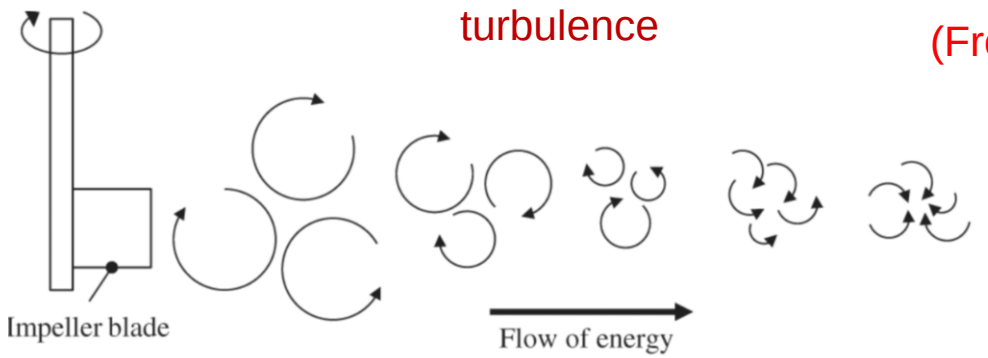




N. Yamamoto's talk, NTU, Taiwan, 18

Inverse energy cascade

- Normal matter : direct cascade



https://doi.org/10.1515/htmp-2016-0043

(From the large scale to the small scale)

standard 3-d magneto-hydrodynamic (MHD) simulations :
disfavor robust supernova explosions (as opposed to the 2-d case)

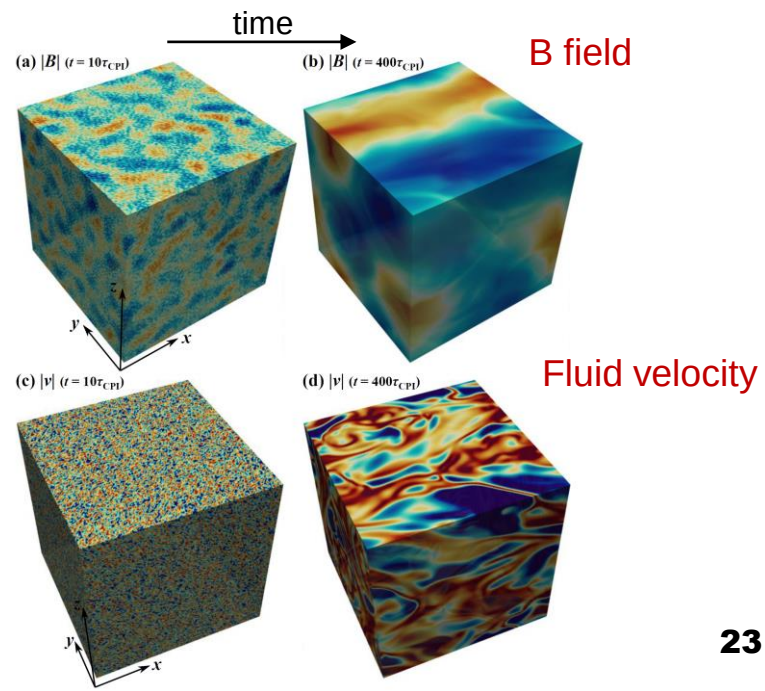
- Chiral matter : inverse cascade (with chiral anomaly)

(From the small scale to the large scale)

could favor supernova explosions

chiral magneto-hydrodynamic (ChMHD) simulations :

Y. Masada, K. Kotake, T. Takiwaki, and N. Yamamoto, PRD 98 (2018) 8, 083018



Chirality transfer & chiral plasma instability

- Chirality (helicity) transfer characterized by helicity conservation :

See e.g. N. Yamamoto, PRD 93, 065017 (2016)

For Weyl fermions : $\partial_\mu j^\mu = -C E^\mu B_\mu$

from CVE & CME

$$\Rightarrow \quad \frac{d}{dt}\int d^3\boldsymbol{x}\left(j^0 + \frac{C}{2}\boldsymbol{A}\cdot\boldsymbol{B}\right) = 0\,, \qquad j^0 = n + \boxed{\xi \boldsymbol{v}\cdot\boldsymbol{\omega} + \xi_B \boldsymbol{v}\cdot\boldsymbol{B}}.$$

$\Rightarrow$ conservation law : $\frac{d}{dt}Q_{\rm tot} = 0\,, \quad Q_{\rm tot} \equiv Q_{\rm chi} + \frac{C}{2}Q_{\rm mag} + \xi Q_{\rm flu} + \xi_B Q_{\rm mix}\,,$

$$Q_{\rm chi} = \int d^3\boldsymbol{x}\; n, \quad Q_{\rm mag} = \int d^3\boldsymbol{x}\; \boldsymbol{A}\cdot\boldsymbol{B}, \quad Q_{\rm flu} = \int d^3\boldsymbol{x}\; \boldsymbol{v}\cdot\boldsymbol{\omega}, \quad Q_{\rm mix} = \int d^3\boldsymbol{x}\; \boldsymbol{v}\cdot\boldsymbol{B}.$$

- Chiral plasma instability (CPI) : Y. Akamatsu and N. Yamamoto, PRL 111, 052002 (2013).

Unstable mode of magnetic fields : $\delta\boldsymbol{B} \propto \exp(i\boldsymbol{k}\cdot\boldsymbol{x} + \sigma t)$

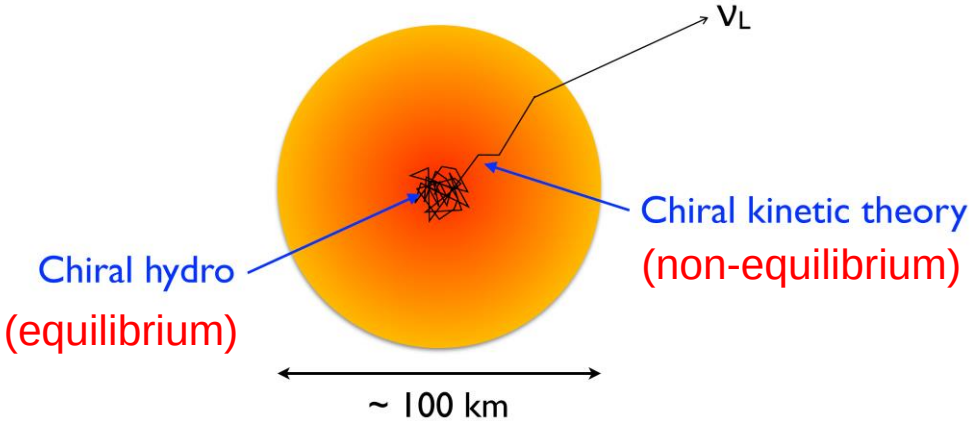
$$\sigma = \eta \boxed{\xi_B} k - \eta k^2 \qquad \text{(small } k\text{)}$$

$\propto$ initial chirality imbalance

Potential applications in astrophysics

- Neutrino transport in core-collapse supernovae
- Weak int. yields intrinsic chiral imbalance

- Neutrinos are out of equilibrium away from the core.



- Near the core : ChMHD (e, N, ν)
- Away from the core : ChMHD (e, N) +CKT (ν) ⟹ chiral radiation hydrodynamics

$$\nabla_\mu T^{\mu\nu}_{\rm rad} + \nabla_\mu T^{\mu\nu}_{\rm mat} = 0$$

(neutrinos)

(electrons, nucleons : equilibrium, non-relativistic)

Chiral radiation transport equation

- CKT for neutrinos : $q \cdot D f^{(\xi)}_{\mathrm{L}q} = (1 - f^{(\xi)}_{\mathrm{L}q})\Gamma^{<}_{(\xi)q} - f^{(\xi)}_{\mathrm{L}q}\Gamma^{>}_{(\xi)q}, \quad n^\mu = \xi^\mu = (1, \mathbf{0}).$

(matter (n, p, e) in equilibrium)

N. Yamamoto & DY, APJ 895 (2020), 1

- Em/ab rates : $\Gamma^{\lessgtr}_{(\xi)q} \approx \Gamma^{(0)\lessgtr}_{q} + \boxed{\hbar\Gamma^{(\omega)\lessgtr}_{q}(q\cdot\omega) + \hbar\Gamma^{(B)\lessgtr}_{q}(q\cdot B)}$

$M_{\mathrm{n}} \approx M_{\mathrm{p}} \approx M$

NR approx., small-momentum transf.

$\omega^\mu \equiv \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}u_\nu(\partial_\alpha u_\beta)$

vorticity & magnetic field corrections :

- incorporating $v \cdot \omega$ & $v \cdot B$ terms
- breaking spherical symmetry & axisymmetry

$\nu^{\mathrm{e}}_{\mathrm{L}}(q) + \mathrm{n}(k) \rightleftharpoons \mathrm{e}_{\mathrm{L}}(q') + \mathrm{p}(k')$

analytic expressions :

$$\Gamma^{(0)>}_{q} \approx \frac{G_{\mathrm{F}}^2}{\pi}(g_{\mathrm{V}}^2 + 3g_{\mathrm{A}}^2)E_\nu^3(1 - f^{(\mathrm{e})}_{q})\left(1 - \frac{3E_\nu}{M_{\mathrm{N}}}\right)\frac{n_{\mathrm{p}} - n_{\mathrm{n}}}{1 - \mathrm{e}^{\beta(\mu_{\mathrm{n}} - \mu_{\mathrm{p}})}}$$

$$\Gamma^{(B)>}_{q} \approx \frac{G_{\mathrm{F}}^2}{2\pi M_{\mathrm{N}}}(g_{\mathrm{V}}^2 + 3g_{\mathrm{A}}^2)E_\nu(1 - f^{(\mathrm{e})}_{q})\left(1 - \frac{8E_\nu}{3M_{\mathrm{N}}}\right)\frac{n_{\mathrm{p}} - n_{\mathrm{n}}}{1 - \mathrm{e}^{\beta(\mu_{\mathrm{n}} - \mu_{\mathrm{p}})}}$$

$$\Gamma^{(\omega)>}_{q} \approx \frac{G_{\mathrm{F}}^2}{2\pi}(g_{\mathrm{V}}^2 + 3g_{\mathrm{A}}^2)E_\nu(1 - f^{(\mathrm{e})}_{q})\left(2 + \beta E_\nu f^{(\mathrm{e})}_{q}\right)\frac{n_{\mathrm{p}} - n_{\mathrm{n}}}{1 - \mathrm{e}^{\beta(\mu_{\mathrm{n}} - \mu_{\mathrm{p}})}}$$

$\Gamma^{(0)>}_{q}$: S. Reddy, M. Prakash, J. M. Lattimer, PRD58:013009,1998

Conclusions & outlook

- ✓ The QKT for fermions from WFs provides a useful theoretical framework to track non-equilibrium spin and chiral transport in phase space.
- ✓ To delineate the dynamical spin polarization of a strange quark traversing wQGP, the QKT of massive fermions is constructed.
- ✓ Similar to massless fermions, photons & gluons may have exotic transport properties under rotation. The QKT of photons/gluons is derived, which can be coupled to the QKT of fermions for the application to wQGP.
- ❖ Details for the collision term in wQCD has to be worked out.

- ✓ Chiral transport of leptons could have potential impacts on the evolution of CCSN.
- ✓ Qualitatively, the chirality imbalance could lead to inverse energy cascade and facilitate the explosions.
- ✓ The chiral transport equation of neutrinos has been constructed from CKT, which should be further applied to the simulations of CCSN.
- ❖ Further simplification or extension of ChRHD will be needed.
- ❖ Other applications of QKT to e.g. condensed-matter physics?

Keio University

1858

CALAMVS GLADIO FORTIOR

Thank you!

WFs in thermal equilibrium

- In global equilibrium : $f_q^{\mathrm{eq}(n)} = (e^g + 1)^{-1}, \quad g = \left(\beta q\cdot u - \bar{\mu} + \frac{\hbar S^{\mu\nu}_{(n)}}{2}\Omega_{\mu\nu}\right), \quad \Omega_{\mu\nu} = \frac{1}{2}\partial_{[\mu}(\beta u_{\nu]}).$

J.-Y. Chen, D. T. Son, M. A. Stephanov, PRL 115, 021601 (2015)

(const. T, μ)

spin-vorticity coupling

Y. Hidaka, S. Pu, DY, PRD 97, 016004 (2018)

chiral anomaly

$$\grave{S}^{<\mu}_{\mathrm{eq}}(q,X) = 2\pi\left[\delta(q^2)\left(q^\mu f_0 + \frac{\hbar\beta}{2}\left(\omega^\mu q\cdot u - u^\mu q\cdot\omega\right) f_0(1-f_0)\right) + \hbar\left(B^\mu q\cdot u - u^\mu q\cdot B\right)\delta'(q^2) f_0\right],$$

fluid vorticity :

$$\omega^\mu \equiv \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}u_\nu(\partial_\alpha u_\beta),$$

$$f_0 = (e^{\beta q\cdot u - \bar{\mu}} + 1)^{-1}.$$

side jump

spin-vorticity coupling

CVE

$$\left(B^\mu(q\cdot u) + q_\perp^\mu \frac{q\cdot B}{q\cdot u}\right) - q^\mu \frac{q\cdot B}{q\cdot u}$$

Berry curvature : kinetic current

magnetic-moment coupling

See also D. Kharzeev, M. Stephanov, H.-U. Yee, PRD 95, 051901 (2017)

$\frac{2}{3} + \frac{1}{3} =$ CME

- Anomalous hydrodynamics (R-handed) : $\partial_\mu T^{\mu\nu} = F^{\nu\rho}J_\rho, \quad \partial_\mu J^\mu = \frac{\hbar}{4\pi^2}(\mathbf{E}\cdot\mathbf{B})$
- Constitutive equations :

$$T^{\mu\nu} = u^\mu u^\nu \epsilon - pP^{\mu\nu} + \Pi^{\mu\nu}_{\mathrm{non}} + \Pi^{\mu\nu}_{\mathrm{dis}}, \quad J^\mu = N_0 u^\mu + v^\mu_{\mathrm{non}} + v^\mu_{\mathrm{dis}}, \quad P^{\mu\nu} = \eta^{\mu\nu} - u^\mu u^\nu.$$

$\mathcal{O}(1)$ $\quad \mathcal{O}(\hbar)$ $\quad \mathcal{O}(1)+\mathcal{O}(\hbar)$ $\quad \mathcal{O}(1)$ $\quad \mathcal{O}(\hbar)$ $\quad \mathcal{O}(1)+\mathcal{O}(\hbar)$

equilibrium : CME/CVE

D. T. Son and P. Surowka, PRL 103, 191601 (2009)

Decompositions of WFs

- Decomposition : D. Vasak, M. Gyulassy, and H. T. Elze, Ann. Phys. 173, 462 (1987).

$$S^< = \mathcal{S} + i\mathcal{P}\gamma^5 + \mathcal{V}_\mu\gamma^\mu + \mathcal{A}_\mu\gamma^5\gamma^\mu + \frac{\mathcal{S}_{\mu\nu}}{2}\sigma^{\mu\nu}, \quad \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu].$$

(pseudo) scalar condensates — vector/axial-charge currents — magnetization

- Currents & EM tensors : $J^\mu_{V/5} = 4\int_q (\mathcal{V}/\mathcal{A})^\mu,\ T^{\mu\nu}_{S/A} = 2\int_q (\mathcal{V}^\mu q^\nu \pm \mathcal{V}^\nu q^\mu),\ \int_q = \int \frac{d^4q}{(2\pi)^4}.$

- Massive fermions : Reducing redundant dof : replacing $\mathcal{S}$, $\mathcal{P}$, and $\mathcal{S}^{\mu\nu}$ in terms of $\mathcal{V}^\mu$ and $\mathcal{A}^\mu$ (10 eqs. → 6 master eqs.).

  e.g. $m\mathcal{P} = -\frac{\hbar}{2}\nabla_\mu\mathcal{A}^\mu \Longrightarrow \partial_\mu J^\mu_5 = \frac{\hbar\mathbf{E}\cdot\mathbf{B}}{2\pi^2} + 2im\langle\bar{\psi}\gamma_5\psi\rangle$

- Massless limit : $\grave{S}^< = \begin{pmatrix} 0 & \sigma^\mu(\mathcal{V}_\mu - \mathcal{A}_\mu) \\ \bar{\sigma}^\mu(\mathcal{V}_\mu + \mathcal{A}_\mu) & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^\mu \grave{S}^<_{\mathrm{L}\mu} \\ \bar{\sigma}^\mu \grave{S}^<_{\mathrm{R}\mu} & 0 \end{pmatrix}$

- LO WFs : $\grave{S}^<_{\mathrm{R/L}\mu} = 2\pi\bar{\epsilon}(q_0)q_\mu f_{\mathrm{R/L}}(q, X),\quad f_{\mathrm{V/A}} \equiv \frac{(f_\mathrm{R} \pm f_\mathrm{L})}{2}.$

2 dynamical variables

Axial kinetic theory

- QKT : modified Boltzmann eqs. tracking charge/spin evolution of a quasi-particle in phase space with quantum corrections.
- Vector/axial-vector $\mathcal{V}^\mu/\mathcal{A}^\mu$: $f_{V/A}(q,X)$ & $\boxed{a^\mu(q,X)}$ spin 4-vector (2 dof)

$$\xrightarrow{m=0} \quad a^\mu = q^\mu, \quad f_{\mathrm{V/A}} \equiv \frac{(f_{\mathrm{R}} \pm f_{\mathrm{L}})}{2}.$$

- Axial kinetic theory : scalar/axial-vector kinetic eqs. (SKE/AKE)
- To include collisions : $\mathcal{V}^\mu \sim \mathcal{O}(\hbar^0)$ and $\mathcal{A}^\mu \sim \mathcal{O}(\hbar)$

with quantum corrections

- Effective QKT for spin : $\boxed{\square^{(n)}\mathcal{A}^\mu} = \boxed{\hat{\mathcal{C}}^\mu_{\mathrm{cl}}} + \boxed{\hbar\hat{\mathcal{C}}^{(n)\mu}_{\mathrm{Q}}}$ DY, K. Hattori, and Y. Hidaka, JHEP 20, 070 (2020)

spin diffusion spin polarization coupled to vector charge

(see also S. Li and H.-U. Yee, PRD100, 056022 (2019))

- The formalism is derived yet the details need to be worked out.
- E.g. Vorticity-induced spin polarization in Nambu–Jona-Lasinio (NJL) model:

Local equilibrium : Z. Wang, X. Guo, P. Zhuang, arXiv:2009.10930

detailed balance ⇨

$$\mathcal{A}^{\mathrm{LE}}_\mu(p) = \mathcal{A}^{\mathrm{LE}(0)}_\mu(p) + \hbar\mathcal{A}^{\mathrm{LE}(1)}_\mu(p) \;=\; -\frac{\hbar}{(2\pi)^3 2E_p}\epsilon_{\mu\nu\sigma\lambda}p^\nu\nabla^\sigma\beta^\lambda f'_{V,\mathrm{LE}}(X,p)$$

(matches the result in global equilibrium)

Chiral Radiation Hydrodynamics

- Energy-momentum conservation : $\nabla_\mu T^{\mu\nu}_{\rm rad} + \nabla_\mu T^{\mu\nu}_{\rm mat} = 0$

  (neutrinos) (electrons, nucleons : equilibrium, non-relativistic)

- On-shell L-handed neutrinos in the inertial frame : $f = f(t_{\rm i}, r, \theta, \phi, E_{\rm i}, \mu_{\rm i}, \bar{\phi}_{\rm i})$

  R. Lindquist, 1966, Annals Phys., 37, 487
  K. Sumiyoshi, & S. Yamada, 2012, Astrophys. J. Suppl.,199, 17

- Transfer equation (CKT) for L-handed neutrinos : $n^\mu = \xi^\mu = (1/c, 0, 0, 0)$

$$\left[\frac{1}{c}\partial_{t_{\rm i}} + \frac{\mu_{\rm i}}{r^2}\partial_r r^2 + \frac{\sqrt{1-\mu_{\rm i}^2}}{r}\left(\frac{\cos\bar{\phi}_{\rm i}}{\sin\theta}\partial_\theta \sin\theta + \frac{\sin\bar{\phi}_{\rm i}}{\sin\theta}\partial_\phi\right) + \frac{1}{r}\partial_{\mu_{\rm i}}(1-\mu_{\rm i}^2) - \frac{\sqrt{1-\mu_{\rm i}^2}}{r}\cot\theta\partial_{\bar{\phi}_{\rm i}}\sin\bar{\phi}_{\rm i}\right] f^{(\xi)}_{\rm Lq}$$

(BF correction from e_L)

$$= \frac{1}{E_i}\left[(1 - f^{(\xi)}_{Lq})\Gamma^{<}_{(\xi)q} - f^{(\xi)}_{Lq}\Gamma^{>}_{(\xi)q}\right], \qquad \Gamma^{<(>)}_{(\xi)q} \equiv \left(q^\nu - \hbar c S^{\mu\nu}_{(\xi)q} D^{(i)}_{q\mu}\right)\Sigma^{<(>)}_{q\nu}$$

(side-jump correction)

- Neutrino absorption/emission & neutrino-nucleon scattering processes :

$$\nu^{\rm e}_{\rm L}(q) + {\rm n}(k) \rightleftharpoons {\rm e_L}(q') + {\rm p}(k')$$ our focus

$$\nu^{\ell}_{\rm L}(q) + {\rm N}(k) \rightleftharpoons \nu^{\ell}_{\rm L}(q') + {\rm N}(k')$$