Non-Unitary TQFTs from 3d $\mathcal{N} = 4$ Rank-0 SCFTs

Myungbo SHIM

Department of Physics
Kyung Hee University, Seoul, Korea

RIKEN iTHEMS-Phys Seminar

mbshim1213@khu.ac.kr

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In collaboration with
Dongmin Gang (SNU), Sungjoon Kim (POSTECH), Kimyeong Lee (KIAS), and Masahito Yamazaki (IPMU)

Today's Goals

Physical Backgrounds and Motivations

In this talk, lots of different subjects are involved to each others: Anyons, Topological QFTs, Modular Tensor Categories, Superconformal field theories in 3d. Some basic concepts/terminologies would help to understanding what's going on.

Understand Our Results: The Proposal and Dictionary

Understand How to

Basically, we computed relevant SUSY partition functions and extracted "modular data" of TQFTs under a special limit called "Degenerate Limit"

Our Main Proposal [Gang, Kim, Lee, MS, Yamazaki 2021]

Non-Unitary TQFTs emerge from degenerate limits of $\mathcal{N}=4$ SCFTs

In a partition function level,

$$\mathcal{Z}^{\mathbb{B}}_{\mathcal{T}_{\text{rank }0}}\left(b^2, m\ (\text{or }\eta), \nu; s\right) \xrightarrow{m\to 0\ (\text{or }\eta\to 1),\ \nu\to\pm 1} \mathcal{Z}^{\mathcal{M}_{g,p}}_{\text{TFT}_{\pm}[\mathcal{T}_{\text{rank }0}]}(s)\ .$$

Classification of $\mathcal{N}=4$ rank-0 SCFTs

In terms of mathematically well-defined TQFTs, we initiate classification of $\mathcal{N}=4$ rank-0 theories which is a kind of blind spots of conformal bootstrap.

Not Only for Partition Functions

We also established a dictionary between non-unitary TQFTs and rank-0 SCFTs

Our Main Result: Dictionaries

$\mathrm{TFT}_\pm[\mathcal{T}_{\text{rank } 0}]$	$\mathcal{T}_{\text{rank } 0}$
$\mathcal{Z}^{\mathcal{M}_{g,p}}_{\mathrm{TFT}_\pm}(s)$	BPS partition function $\mathcal{Z}^{\mathbb{B}}_{\mathcal{T}_{\text{rank } 0}}\big\|_{\nu\to\pm1,m=0}(s)$ with (topology of $\mathbb{B}$) $=\mathcal{M}_{g,p}$
Spin or non-spin	Next Slide
Rank N	Witten index
Simple objects	Bethe vacua $\{\vec{z}_\alpha\}_{\alpha=0}^{N-1}$ or BPS loop operators $\{\mathcal{O}^\pm_\alpha(\vec{z})\}_{\alpha=0}^{N-1}$
$(S^\pm_{0\alpha})^{-2}$	$\mathcal{H}_\alpha(m=0,\nu\to\pm1;s=-1)$
$T^\pm_{\alpha\beta}$ (only for non-spin)	$\delta_{\alpha\beta}(\mathcal{F}_\alpha/\mathcal{F}_{\alpha=0})\big\|_{\nu\to\pm1,m=0}$
$(T^2)^\pm_{\alpha\beta}$	$\delta_{\alpha\beta}(\mathcal{F}_\alpha/\mathcal{F}_{\alpha=0})^2\big\|_{\nu\to\pm1,m=0,s=-1}$
$S^\pm_{00}$	$\left\|\mathcal{Z}^{S^3_b}_{\mathcal{T}_{\text{rank } 0}}(m=0,\nu\to\pm1)\right\|$
$W^\pm_\beta(\alpha)$	$\mathcal{O}^\pm_\alpha(\vec{z}_\beta)\|_{\nu\to\pm1,m=0}$
$\max_\alpha(-\log\|S^\pm_{0\alpha}\|)$	F (three-sphere free energy)

0 : Why 3d Physics Interesting and Distinct?

Spacetime Symmetries in 3d

Parity in $\mathbb{R}^{1,2}$ and $\mathbb{R}^{1,3}$

Spatial inversion in 3d is an element of the rotation group $\mathrm{SO}(2)$. Parity is defined as a mirror inversion in 3d.

$$
\begin{aligned}
\mathcal{P}_3 : &\qquad (x, y) \to (x, -y) \qquad \text{or} \qquad (x, y) \to (-x, y) \\
\mathcal{P}_4 : &\qquad (x, y, z) \to (-x, -y, -z) \\
\mathcal{P}_3^E : &\qquad (x, y, z \equiv -it) \to (x, y, -z)
\end{aligned}
$$

In Euclidean signature, parity can be identified to time reversal in Lorentzian signature

Frequently Referred Spacetime 3-Manifolds

Flat Spacetimes: $\mathbb{R}^{1,2}$, $\mathbb{R}^3$, and $S^1_\beta \times \mathbb{R}^2$

Usual backgrounds for particle/condensed matter physics on an infinite plane in Lorentzian and Euclidean Signatures. Compactifying time directions lead us a thermal theories.

Curved Spacetime: S^3, $\mathbb{R} \times S^2$, $S^1 \times S^2$

Superconformal theories on S^3 are usually considered in a context of AdS/CFT. $\mathbb{R} \times S^2$ appeared as a background of radial quantization of conformal field theories on flat spacetimes. $S^1 \times S^2$ appeared in radially quantized thermal CFT on flat spacetime.

Summaries on Some 3-Manifolds

Spacetime Symmetries and Some Relevant Physical Quantities

	$\mathbb{R}^{1,2}$	$S^1 \times S^2$	S^3
Isometry	$\mathrm{ISO}(1{,}2)$	$\mathrm{U}(1) \times \mathrm{SO}(3)$	$\mathrm{SO}(4)$
Num. of Gens.	6	4	6
Discrete Symmetry	$\mathcal{C}$, $\mathcal{P}$, $\mathcal{T}$	$\mathcal{C}$, $\mathcal{P} = \mathcal{T}$	$\mathcal{C}$, $\mathcal{P} = \mathcal{T}$
SUSY Indices	Witten Index	Superconformal Index	Free Energy

Table: Witten and Superconformal indices are captured in terms of topologically twisted index which is defined on Seifert fibration with the first Chern number p, $S^1 \times_p \Sigma_{\mathfrak{g}}$ where $\Sigma_{\mathfrak{g}}$ is a Riemann surface of genus-$\mathfrak{g}$

Appetizer: Abelian Gauge Theory

Maxwell Theory in 3d: What is different?

Most familiar gauge theory is the Maxwell Theory. Action and field equations are the same as 4-dimension. However, physics is little bit different.

$$E^i = F^{0i}, \qquad B = \epsilon^{012} F_{12}$$

Magnetic field is no longer vector field.

Game Changer: Topological Current

Bianchi Identity can be considered as continuity equation for topological current

$$dF = 0 \leftrightarrow \partial_\mu \left(\frac{1}{4\pi} \epsilon^{\mu\nu\rho} F_{\nu\rho} \right) = \partial_\mu J^\mu_{top}$$

In 4-dimension, this lead us to a notions of 1-form symmetry and topological loop operator.

Magnetic Monopole in 3d

Charge of Topological Current

$$Q_{top} = \int d^2x\, J^0_{top} = \int d^2x\, \frac{1}{2\pi}\epsilon^{012}F_{12} = \frac{1}{2\pi}\int d^2x\, B$$

Topological Charges act on a Disorder Operator: Monopole Operator

There is no field which is charged under $\mathrm{U}(1)_{top}$ even after the theory coupled with matter fields. As in 't Hooft line or loop operators in 4d, it is **disorder operator** rather than local operator. In 3d, the charge is 0-form, the charged object is a particle-like operator. It is called **monopole operator**

BF Theory and Chern-Simons Theory

Minimal Coupling with Background Gauge Field for Topological Current

Consider minimal coupling to "background" gauge field B to topological current.

$$\mathcal{S}_{BF} = -\int d^3x J^\mu B_\mu = -\int d^3x \frac{1}{4\pi}\epsilon^{\mu\nu\rho}F_{\nu\rho}B_\mu = -\frac{1}{2\pi}\int B\wedge F$$

This action is also called as BF action. It is an off-diagonal Chern-Simons term.

Here Comes New Player: Chern-Simons Action

For Abelian gauge field, CS action is written as

$$\mathcal{S} = \frac{k}{8\pi}\int d^3x \epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho = \frac{k}{4\pi}\int A\wedge dA, \qquad k\in\mathbb{Z}$$

This term is parity odd, and its variation is a total derivative.

$\mathrm{U}(1)_k$ Chern-Simons-Matter Theory

Non-vacuum Equations of Motion

Including current/source term, the Abelian CS theory is

$$\mathcal{S} = \int d^3x \left(\frac{k}{8\pi} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho - A_\mu J^\mu \right)$$

Variation yields

$$dF = \partial_\mu J^\mu = 0, \qquad \frac{k}{4\pi} \epsilon^{\mu\nu\rho} F_{\nu\rho} = J^\mu$$

Interpretation of Equations of Motion

Contrary to the usual electromagnetism, in CS theory, electric charges source magnetic fluxes, and electric currents source electric fields.

$$\epsilon^{012} F_{12} = \frac{4\pi}{k} J^0 \to B = \frac{4\pi}{k} Q_e, \qquad \epsilon^{ij0} F_{j0} = \frac{4\pi}{k} J^i \to E^i = \epsilon_{ij} J^j$$

Anyons in CSM Theories

Magnetic Fluxes attached at an Electric Charge

Every charges has magnetic flux at a place where the charges locate. From this property, rotating a charge around other charge is the same as rotating around an area where magnetic fluxes exist: Aharonov-Bohm effect.

Ahronov-Bohm Phase or Exchange Phase $\rightarrow$ Fractional(or Braiding) Statistic

Exchange of two bosons or fermions leads a phase factor ± 1 depending on their (topological) spin s.

$$\text{Exchange Phase} = e^{\pi i(2s)} = \begin{cases} 1 & (s \in \mathbb{Z}) \\ -1 & (s \in \mathbb{Z} + \frac{1}{2}) \\ e^{2\pi i s} & \text{Otherwise} \Leftrightarrow \text{Anyons} \end{cases}$$

A particle with non-trivial phase factor after exchange is called anyon. They belong to infinite dimensional representation of 2+1d Poincaré algebra [Jackiw, Nair 1991] with non-(half)-integer spin. Anyon means particles with "any" spins.

1 : Introduction to Superconformal Theories in 3d

$\mathcal{N} = 2$ Supersymmetry: 4 Real Supercharges

$\mathrm{SO}(2) \cong \mathrm{U}(1)$ R-Symmetry

In superconformal theory, R-charges are the same as conformal dimensions due to the algebra.

Matter Contents

- V Vector multiplet: A_μ, λ and $\bar{\lambda}$, σ, and D, conf. dimension $\Delta_V = 1$
- Φ Chiral multiplet: ϕ, ψ, and F, conf. dimension $\Delta_\Phi = 1/2$
- $\bar{\Phi}$ Anti-Chiral multiplet: $\bar{\phi}$, $\bar{\psi}$, and $\bar{F}$, conf. dimension $\Delta_{\bar{\Phi}} = 1/2$
- Σ Linear multiplet: $\Sigma = \pm\frac{i}{2}\epsilon^{\alpha\beta}\bar{D}_\alpha(e^{-V}D_\beta e^V) = \pm\frac{i}{2}\epsilon^{\alpha\beta}\bar{D}_\alpha D_\beta V$ up to sign convention. It is relevant to monopole operators.

$\mathcal{N} = 4$ Supersymmetry: 8 Real Supercharges

$\mathrm{SO}(4) \cong \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ R-Symmetry

Let R and R' be two Cartans of $\mathrm{SU}(2)_L$ and $\mathrm{SU}(2)_R$ resp, and let R_ν and A be two Cartans of $\mathrm{SO}(4)$ R-symmetry. (ν: mixing parameter)

$$R_{\mathrm{IR}} = R_{\nu=0} = R + R', \qquad A = R - R', \qquad R_\nu = R + R' + \nu A,$$

In $\mathcal{N} = 2$ language, R_ν is $\mathrm{U}(1)$ R-symmetry and A is $\mathrm{U}(1)$ flavor symmetry.

Matter Contents

- V Vector multiplet: $\mathcal{N} = 2$ vector $V + \mathcal{N} = 2$ chiral multiplet Φ_{adj} in adjoint representation of gauge group
- h Hypermultiplet: $\mathcal{N} = 2$ chiral Φ + anti-chiral multiplet $\bar{\Phi}$ in R and $\bar{R}$ representations.
- $h_{\frac{1}{2}}$ Half-hyper: If the gauge group is pseudo-real, there is a relation between Φ and $\bar{\Phi}$. Then, the d.o.f is just a half of h.

Moduli Space of SUSY Theory

Classical Vacuum Moduli Space

All the potentials are encoded in a superpotential $\mathcal{W}$, and vacuum is defined $dV/d\phi = 0$.

$$V = \sum_{i=1}^{N_f} \left| \frac{d\mathcal{W}}{d\phi_i} \right|^2, \qquad \frac{dV}{d\phi_1} = \cdots \frac{dV}{d\phi_{N_f}} = 0$$

There could be multiple families of solutions. Each families are called "Branch".

Classical SUSY Vacuum Moduli Space

SUSY invariant vacuum is global minimum with zero value. The vacuum equations are linearized. They are called F/D-Flatness conditions

$$V = 0 \qquad \Leftrightarrow \qquad F_a = 0 \text{ and } D = 0$$

Branches of SUSY Moduli Spaces

1: Coulomb Branch: $G \to \mathrm{U}(1)^{k \leq \mathrm{rank}(G)}$. Just Maxwell theories(Coulomb forces)

2: Higgs Branch: $G \to H$. There occurs Higgs mechanism.

3: Mixed Branch: $G \to \mathrm{U}(1)^{k \leq (\mathrm{rank}(G) - \mathrm{rank}(H'))} \times H'$. Hybrid of those two.

Rank of SCFTs

Definition

The terminology "Rank" of SCFT means that maximum complex dimensions of vacuum moduli space in Coulomb and Higgs branch.

$$\mathrm{rank} = \mathsf{max}(\dim(\mathcal{M}_{\mathrm{Coulomb}}), \dim(\mathcal{M}_{\mathrm{Higgs}}))$$

$$\text{Rank-0 Theory} = \text{Theory with no Higgs and Coulomb branches}$$

Some Properties of Rank-0 Theory

1: Coulomb/Higgs branch operators are charged under $\mathrm{SU}(2)_{L/R}$ of $\mathrm{SO}(4)$ R-symmetry. Flavor current multiplets includes Coulomb or Higgs branch operators

2: There are no flavor commuting with R-symmetry for Rank-0 Theory.

3: SUSY should be no more than $\mathcal{N} = 5$.

$\mathcal{N} = 4$ Rank-0 SCFTs in 3d

$\mathcal{N} = 4$ Rank-0 SCFT

Theories invariant under $\mathcal{N} = 4$ superconformal transformation, and the theories have no Coulomb and Higgs branches.

Degenerate Limit: $\nu = \pm 1$

Degenerate limit is defined non-trivial R-charge mixing parameter $\nu = \pm 1$. For $\nu = 1$, SCI gives Hilbert series of Coulomb branch. $\nu = -1$ corresponds to Higgs branch. They are equivalent by 3d mirror symmetry.

Target Supersymmetric Partition Functions

1 : S^3 partition function $\rightarrow F_{S^3} = \log \mathcal{Z}_{S^3}$ [Jafferis 2010]

2 : S_b^3 Squashed three-sphere partition function

3 : $S^1 \times S^2$ partition function $\Leftrightarrow$ Superconformal Index (SCI) [Kim 2009, Imamura, Yokoyama 2011, Kapustin, Willett 2011]

4 : $S^1 \times \Sigma_{\mathfrak{g}}$ partition function $\Leftrightarrow$ Topologically Twisted Index ($\mathfrak{g}$=0$\rightarrow$ SCI)

S_b^3 Partition Function: Super Versatile Tool

SUSY Localization on Squashed Three Sphere

Field contents + charge table $\rightarrow$ S_b^3 partition function by localization [Hama, Hosomichi, Lee 2011]

Method of Bethe Vacua

From the squashed three-sphere partition function, one can consider twisted superpotential and its saddle points, Bethe Vacua, from S_b^3 partition function. Then, one can construct handle gluing $\mathcal{H}$, fibering $\mathcal{F}$ operators, and general topologically twisted indices [Closset, Kim, Willett 2017, 2018]

Building Blocks of SUSY Partition Function - 1

Superconformal Index: SUSY Partition Function on $S^1 \times S^2$

Generic structure of SCI with a gauge group G is the below.

$$\begin{aligned}&\mathcal{I}^{\text{sci}}(q,\eta,\nu=0;s=1)\\&=\sum_{\mathrm{m}}\oint_{|a_i|=1}\left(\prod_{i=1}^{\text{rank}G}\frac{da_i}{2\pi i a_i}\right)\Delta_G(\mathrm{m},\mathrm{a};q)q^{\epsilon_{\mathbf{0}}(\mathfrak{n})}\mathcal{I}_0^{cs}(\mathrm{m},\mathrm{a})\text{P.E.}[f_{\text{single}}(q,\mathrm{a},\eta;\mathrm{m})]\,.\end{aligned}$$

Going to Degenerate Limit $\nu=\pm 1$: Non-superconformal R-charge

$$\mathcal{I}^{\text{sci}}(q,\eta,\nu;s=\pm 1)=\mathcal{I}^{\text{sci}}(q,\eta,\nu=0;s=\pm 1)\Big|_{\eta\to\eta(\pm q^{\frac{\nu}{2}})}$$

Localization Recipes

Each building blocks in the above formula are obtained by localization recipes with matter contents [Kapustin, Willett 2011, Imamura, Yokoyama 2011]

Building Blocks of SUSY Partition Function - 2

$\mathcal{Z}_{S_b^3}$: SUSY Partition Function on S_b^3

Generic structure of $\mathcal{Z}_{S_b^3}$ with a gauge group G is the below.

$$\mathcal{Z}^{S_b^3}(b, m, \nu) = \int \left(\prod_{i=1}^{\text{rank}(G)} \frac{dZ_i}{\sqrt{2\pi\hbar}} \right) \Delta_G(\mathrm{Z};\hbar)\mathcal{I}_\hbar(\mathrm{Z}, m, \nu) \ , \quad \hbar := 2\pi i b^2 \ .$$

$\{Z\}$ parametrize Cartan subalgebra of G.

Localization Recipes

Each building blocks in the above formula are obtained by localization recipes with matter contents [Jafferis 2010, Hama, Hosomichi, Lee 2011]

Bethe Vacua Recipes

$\mathcal{Z}_{S^3_b}$, Twisted Superpotential $\mathcal{W}$, and Perturbative Expansions around Saddles

$$\mathcal{Z}^{S^3_b}(b,m,\nu) = \int \left(\prod_{i=1}^{\mathrm{rank}(G)} \frac{dZ_i}{\sqrt{2\pi\hbar}} \right) \exp[\log \mathcal{I}_\hbar(\mathrm{Z},m,\nu)] \,, \quad \hbar := 2\pi i b^2 \,.$$

$$\mathcal{W}_n \,:\, \log \mathcal{I}_\hbar(\vec{Z},m,\nu) \xrightarrow{\hbar\to 0} \sum_{n=0}^{\infty} \hbar^{n-1}\mathcal{W}_n(\vec{Z},n,\nu) \,.$$

Around the saddles,

$$\mathcal{Z}^{S^3_b}(b,m,\nu) = |\mathrm{Weyl}(G)| \times \int \prod_{i=1}^{\mathrm{rank}(G)} \frac{d(\delta Z_i)}{\sqrt{2\pi\hbar}} \exp\left(\frac{1}{\hbar}\mathcal{W}_0^{\vec{n}_\alpha}(\vec{Z}^\alpha + \delta\vec{Z},m,\nu) + \sum_{n=1}^{\infty} \hbar^{n-1}\mathcal{W}_n(\vec{Z}+\delta\vec{Z},m,\nu) \right) \xrightarrow{\hbar\to 0} \exp\left(\sum_{n=0}^{\infty} \hbar^{n-1}\mathcal{S}_n^\alpha(m,\nu) \right)$$

Bethe Vacua Recipes - 2

Bethe Vacua: Saddle Points

$$\frac{\left\{\vec{z}\ :\ \left(\exp(\partial_{Z_i}\mathcal{W}_0)\big|_{\vec{Z}\to\log\vec{z}}\right)=1\ ,\ w\cdot\vec{z}\neq\vec{z}\ \ \forall\,\text{non-trivial}\ w\in\mathrm{Weyl}(G)\right\}_{i=1}^{\mathrm{rank}(G)}}{\mathrm{Weyl}(G)}$$

Handle Gluing $\mathcal{H}$ and Fibering $\mathcal{F}$ Operators

$$\mathcal{H}_\alpha(\eta,\nu;s=-1)=e^{i\varphi}\exp\left(-2\mathcal{S}_1^\alpha(m,\nu)\right)\big|_{m=\log\eta}\ .$$
$$\mathcal{F}_\alpha(\eta,\nu;s=-1)=\exp\left(\frac{\mathcal{S}_0^\alpha(m,\nu)}{2\pi i}\right)$$

Bethe Vacua Recipes - 3

Twisted Index on $S^1 \times_p \Sigma_\mathfrak{g}$

$$\mathcal{Z}^{\mathcal{M}_{g,p}}(m,\nu,s) = \sum_{\vec{z}_\alpha \text{ : Bethe-vacua}} (\mathcal{H}_\alpha(\eta = e^m, \nu; s))^{g-1} (\mathcal{F}_\alpha(m,\nu;s))^p$$

Twisted Index on $S^1 \times \Sigma_\mathfrak{g}$

$$\mathcal{Z}^{\mathcal{M}_{g,0}}(m,\nu,s) = \mathcal{Z}^{S^1 \times \Sigma_\mathfrak{g}}(m,\nu,s) = \sum_{\vec{z}_\alpha \text{ : Bethe-vacua}} (\mathcal{H}_\alpha(\eta = e^m, \nu; s))^{g-1}$$

2 : Introduction to Anyons/TQFTs in 3d

Anyon is a distinct feature of 3d

Fractional Spin-Statistics

Bosons and Fermions have (half-)integer phase factors after exchange of two identical particles at X and Y in spacetime.

$$|(Y,X)\rangle_{b,f} = e^{2\pi i s}|(X,Y)\rangle = \pm|(X,Y)\rangle_{b,f}, \qquad s \in \frac{\mathbb{Z}}{2}$$

Anyons have non-trivial phase factors with fractional spins.

$$|(Y,X)\rangle_a = e^{2\pi i s_a}|(X,Y)\rangle_a, \qquad s_a \in \mathbb{Q}$$

Bridges between Physics and Mathematics

Anyon physics has non-trivial fusion rules between anyonic operators and ground states structure in its Hilbert space. These are encapsulated in mathematically well-defined object, Modular Tensor Category.

Why Are Anyons Topological?

Statistics is NOT completely arbitrary

Formal inspection of path-integral of Anyons lead us that statistics are constrained by topology of 2d spaces, *i.e.* $\pi_1(\Sigma_\mathfrak{g})$

Fractional Statistics on Torus

In a case of torus, statistics $2s$ are a rational number labeled by two integer p/q from $\pi_1(\mathbb{T}^2) \cong \mathbb{Z}^2$

Anyons and Chern-Simons Theory

Abelian Examples

1: Integer QHE: (half-) Integer particle statistics

2: FQHE: Fractional charges and particle statistics[Tsui, Stormer, and Gossard 1982, Halperin 1984, Wilczek, Arovas, and Schrieffer 1985]

Other examples

3: Non-Abelian Anyons: Promising hints but no experimental evidence yet [Moore, Read 1991, Wen 1991]

4: Toric Code Model: Topological quantum computing [Kitaev 1997] and Simplest example of topological order [Read, Sachdev 1991, Wen 1991]

Anyons and Chern-Simons Theory

Holography: Rational CFT on boundary and TQFT in 3d

1: Rational conformal field theories are studied by Seiberg and Moore. Key data are called as Moore-Seiberg data including modular S and T matrices, fusion rings of primary operators [Moore, Seiberg 1989]

2: Verlinde found a connection between modular S matrix and fusion rings [Verlinde 1988]

3-a: Reshetikhin and Turaev constructed 3d topological field theories of which boundary theory is RCFTs. These two theory has the same Moore-Seiberg data[Reshetikhin, Turaev 1991, Turaev 1992].

3-b: Constructions in terms of 2d RCFT are done by Fjelstad–Fuchs–Runkel–Schweigert [Fjelstad, Fuchs, Runkel, Schweigert 2005]

4: These modular data are turned out to be a well-defined mathematical subject, *Modular Tensor Category*.

Anyons and Chern-Simons Theory

Recent Novel Approaches to Anyon Physics from 3d

1: $\mathrm{SL}(N, \mathbb{C})$ Chern-Simons Theories on non-hyperbolic 3-manifolds [Cho, Gang, Kim 2020]

2: Topological Twist of 3d $\mathcal{N} = 4$ Theory and Rozansky-Witten Theory [Gukov, Hsin, Nakajima, Park, Pei, Sopenko 2020]

3: Degenerate limit of $\mathcal{N} = 4$ Rank-0 SCFTs [Gang, Kim, Lee, MS, Yamazaki 2021] ✓ **Today's agenda**

Ground States of TQFTs

Anyon from (1+2)d TQFTs

In TQFT, there are no local operators, but loop operators exist. For examples, Chern-Simons theories have Wilson loop operators, Toric codes have loop operators on spin-lattice.

Anyon and Ground State Degeneracy

Anyon is not a local excitation. It is a kind of zero-mode excitation. Thus counting anyonic states yields ground state degeneracy.

Anyon and $\mathcal{P}$, $\mathcal{T}$ Anomaly in 3d

Anyon statistics are closely related to broken discrete symmetry such as parity and time translation.

Ground States of TQFTs

Anyons on Torus

As a convention, anyons are labelled on torus Hilbert space $\mathcal{H}(\mathbb{T}^2)$. We take a basis

$$\text{Bases of } \mathcal{H}(\mathbb{T}^2) = \{|\alpha\rangle \equiv \mathcal{O}^B_\alpha|0\rangle | \mathcal{O}_0 = 1, \alpha = 0 \sim N-1\},$$

where B is a generator of 1st integral homology $H_1(\mathbb{T}^2, \mathbb{Z})$, so called B-cycle.

Modular S, T Matrices

Modular group of a torus $\mathrm{SL}(2,\mathbb{Z})$ has generators $\mathbb{S}$ and $\mathbb{T}$ acting on the Hilbert space, matrix elements are obtained as

$$S_{\alpha\beta} = \langle\alpha|\mathbb{S}|\beta\rangle, \qquad T_{\alpha\beta} = \exp\left(\frac{2\pi i c_{2d}}{24}\right)\langle\alpha|\mathbb{T}|\beta\rangle$$

Fusion Rules of Anyons

Basic Operators and Fusions

Basic operators *w.r.t.* A and B cycles act on the Hilbert space as

$$\mathcal{O}^A_\beta|\alpha\rangle = W_\beta(\alpha)|\alpha\rangle, \qquad \mathcal{O}^B_\beta|\alpha\rangle = \mathcal{O}^B_\beta\mathcal{O}^B_\alpha|0\rangle = \sum_{\gamma=0}^{N-1} N^\gamma_{\alpha\beta}|\gamma\rangle,$$

where $W_\beta(\alpha) = S_{\alpha\beta}/S_{\alpha 0}$.

Fusion Rings and Verlinde Formula

Fusions of operators form a commutative ring, and the fusion coefficient can be obtained from S matrix by Verlinde formula

$$\mathcal{O}^B_\beta\mathcal{O}^B_\alpha = \sum_{\gamma=0}^{N-1} N^\gamma_{\alpha\beta}\mathcal{O}^B_\gamma, \qquad N^\gamma_{\alpha\beta} = \sum_{\delta=0}^{N-1} \frac{S_{\delta\alpha}S_{\delta\beta}\bar{S}_{\delta\gamma}}{S_{0\delta}}$$

3 : Introduction to Modular Tensor Categories

Modular Tensor Category in Physics

Key Mathematical Structure of Anyons, TQFTs, and RCFTs

All the core information of Anyons/TQFTs/RCFTs are encapsulated in Modular Tensor Category. Mainly, invariants of MTC are physical:

1: **Simple Objects** $\Leftrightarrow$ Anyons, Loop Operators, or Chiral Primaries

2: **Modular Data** $\Leftrightarrow$ Modular Transformations of 2-Torus $\mathbb{T}^2 = \Sigma_{\mathfrak{g}=1}$

3: **Grothendick Semiring** $\Leftrightarrow$ Fusion Rules of Anyons/Loop Operators/Chiral Primaries

4: **Rank of MTC** $\Leftrightarrow$ The number of simple objects or ground state degeneracy(GSD) on $\mathbb{T}^2$

Some Definitions with Relevant Keywords

Road to Modular Tensor Category

MTC is axiomatized by Turaev [Turaev 1992] based on Moore-Seiberg data and Braided Tensor Category [Joyal, Street 1993]. *c.f.* [Bruillard, Ng, Rowell, Wang 2014]

Formal Definition of MTC - 1

MTC is a braided spherical fusion category with the non-degenerate braidings.

1: **Fusion Category**: Abelian $\mathbb{C}$-Linear semi-simple rigid monoidal category with simple objects with 1, finite dimensional morphism space and finitely many isomorphism classes of simple objects.

This is a foundational base for fusions of Anyons(simple objects)

Some Definitions with Relevant Keywords

Formal Definition of MTC - 2

MTC is a braided spherical(pivotal) fusion category with the non-degenerate braidings.

2: **Braiding**: A Family of isomophisms of the fusion category which satisfies the hexagonal diagram. This structure reflects braidings of anyon physics.

3: **Pivotality**: Existence of an isomorphism of monoidal functors in the fusion category. One can define left-/right pivotal trace of an endomorphism in the fusion category

4: **Spherical** is a case that left and right pivotal trace coincide each other for all the endomorphism.

Some Definitions with Relevant Keywords

Formal Definition of MTC - 3

MTC is a braided spherical(pivotal) fusion category with the non-degenerate braidings.

5: **Twist/Ribbon structure** is $\mathbb{C}$-Linear automorphism θ of $Id_{\mathcal{C}}$. With Ribbon structure, braided spherical fusion categories become premodular category. In this stage, one can define S-matrix.

6: **Modular Tensor Category**: If one can define non-singular S-matrix of the premodular(ribbon) category, it is a modular tensor category.

Data of Interests

Modular S-Matrix

Non-singular S-matrix is one of our main target. The first low/column of S matrix has entries as categorical dimensions of simple objects.

Modular T-Matrix

Modular T-matrix contains topological spins of simple objects.

Unitarity of MTC

Unitarity $S^{-1} = S^\dagger$

Identity operator has minimum categorical dimensions $\{S_{0\alpha}\}$

$$|S_{00}| \leq |S_{0\alpha}|$$

Non-Unitary Condition

Identity operator has not minimum categorical dimensions $\{S_{0\alpha}\}$

$$|S_{00}| > |S_{0\alpha}|$$

Classification of Modular Tensor Categories

Classification by Rank

1: All 35 Unitary Modular Tensor Categories ($r \leq 4$) up to ribbon tensor equivalences are classified [Rowell, Stong, Wang 2007]

2: Rank 5 Modular Tensor Categories [Bruillard, Ng, Rowell, Wang 2015]

3: Partial list of primitive UMTC of $r = 7, 8, 9$ [Wen 2015]

4: Classification of Fermionic MTC [Bruillard, Galindo, Hagge, Ng, Plavnik, Rowell, Wang 2016]

4 : Non-Unitary TQFTs/$\mathcal{N} = 4$ Rank-0 SCFTs Correspondence

Our Main Proposal [Gang, Kim, Lee, MS, Yamazaki 2021]

Non-Unitary TQFTs emerge from degenerate limits of $\mathcal{N}=4$ SCFTs

In a partition function level,

$$\mathcal{Z}^{\mathbb{B}}_{\mathcal{T}_{\text{rank }0}}\left(b^2, m\ (\text{or } \eta), \nu; s\right) \xrightarrow{m\to 0\ (\text{or } \eta\to 1),\ \nu\to\pm 1} \mathcal{Z}^{\mathcal{M}_{g,p}}_{\text{TFT}_\pm[\mathcal{T}_{\text{rank }0}]}(s)\ .$$

Classification of $\mathcal{N}=4$ rank-0 SCFTs

Using mathematically well-defined TQFTs, we initiate classification of $\mathcal{N}=4$ rank-0 theories which is a kind of blind spots of conformal bootstrap.

Not Only for Partition Functions

We also established a dictionary between non-unitary TQFTs and rank-0 SCFTs

Main Result: Dictionaries

$\mathrm{TFT}_\pm[\mathcal{T}_{\text{rank } 0}]$	$\mathcal{T}_{\text{rank } 0}$
$\mathcal{Z}^{\mathcal{M}_{g,p}}_{\mathrm{TFT}_\pm}(s)$	BPS partition function $\mathcal{Z}^{\mathbb{B}}_{\mathcal{T}_{\text{rank } 0}}\big\vert_{\nu\to\pm1,m=0}(s)$ with (topology of $\mathbb{B}$) $=\mathcal{M}_{g,p}$
Spin or non-spin	Next Slide
Rank N	Witten index
Simple objects	Bethe vacua $\{\vec{z}_\alpha\}_{\alpha=0}^{N-1}$ or BPS loop operators $\{\mathcal{O}^\pm_\alpha(\vec{z})\}_{\alpha=0}^{N-1}$
$(S^\pm_{0\alpha})^{-2}$	$\mathcal{H}_\alpha(m=0,\nu\to\pm1;s=-1)$
$T^\pm_{\alpha\beta}$ (only for non-spin)	$\delta_{\alpha\beta}(\mathcal{F}_\alpha/\mathcal{F}_{\alpha=0})\big\vert_{\nu\to\pm1,m=0}$
$(T^2)^\pm_{\alpha\beta}$	$\delta_{\alpha\beta}(\mathcal{F}_\alpha/\mathcal{F}_{\alpha=0})^2\big\vert_{\nu\to\pm1,m=0,s=-1}$
$S^\pm_{00}$	$\left\vert\mathcal{Z}^{S^3_b}_{\mathcal{T}_{\text{rank } 0}}(m=0,\nu\to\pm1)\right\vert$
$W^\pm_\beta(\alpha)$	$\mathcal{O}^\pm_\alpha(\vec{z}_\beta)\vert_{\nu\to\pm1,m=0}$
$\max_\alpha(-\log\vert S^\pm_{0\alpha}\vert)$	F (three-sphere free energy)

Spin/Non-spin TQFT from $\mathcal{T}_{\text{rank0}}$

Spin(Fermionic) TQFT from SCI

1: $R_{\nu=\pm 1} + 2j_2 = 1 \text{ mod } 2$

2: if $\mathcal{I}^{\text{sci}}(q, \eta, \nu = \pm 1; s)$ contains $q^{\frac{1}{2}(\text{odd integer})}$ terms .

Non-Spin(Bosonic) TQFT from SCI

1: $R_{\nu=\pm 1} + 2j_2 = 0 \text{ mod } 2$

2: if $\mathcal{I}^{\text{sci}}(q, \eta, \nu = \pm 1; s)$ contains **no** $q^{\frac{1}{2}(\text{odd integer})}$ terms .

Application: Bound on F_{S^3} and $S_{0\alpha}$

Interesting Dictionary on F

$$F_{S^3} = \max_{\alpha}(-\log|S^{\pm}_{0\alpha}|) \Rightarrow F_{S^3} > \frac{1}{2}\log\left(r = \text{Witten Index}\right)$$

This comes from the fact

$$\sum_{\alpha=0}^{N-1}(S_{0\alpha})^2 = 1 \rightarrow \min_{\alpha}|S_{0\alpha}| < \frac{1}{\sqrt{N}}$$

Physics: Lower Bounds of Genuine Measure of D.O.F. F_{S^3}

Mathemtatics: Upper Bound for Minimum Categorical Dimensions

Minimal Rank-0 Theory $\mathcal{T}_{\text{min}}$

UV $\mathcal{N} = 2$ $\mathrm{U}(1)_{k=-3/2}$ with a chiral multiplet of charge 1

In IR, supersymmetries are enhanced to $\mathcal{N} = 4$.

What is minimal? Degrees of Freedom

At the first shot, it seems that a theory with a free chiral multiplet has minimal degrees of freedom.

Comparison of central charge [Gang, Yamazaki 2018]

Central charge is one of measures of degrees of freedom of conformal theories.

$$\frac{C_T(\mathcal{T}_{\text{min}})}{C_T(\text{free theory with single } \Phi)} \simeq 0.992549$$

Lee-Yang TQFT from Degenerate Limit of $\mathcal{T}_{\text{min}}$

Hints of TQFT: SCI

SCI with conformal R-charges is

$$\mathcal{I}^{\text{sci}}_{\mathcal{T}_{\text{min}}}(q, u, \nu = 0; s = 1) = 1 - q + \left(\eta + \frac{1}{\eta}\right) q^{3/2} - 2q^2 + \cdots \ .$$

Consider non-trivial R-charge mixing, $\nu = \pm 1$, SCI is

$$\begin{aligned} &\mathcal{I}^{\text{sci}}_{\mathcal{T}_{\text{min}}}(q, \eta, \nu, s = 1)\Big|_{\nu \to \pm 1} \\ &= 1 + \left(-1 + \eta^{\mp 1}\right) q + \left(-2 + \eta + \frac{1}{\eta}\right) q^2 + \left(-2 + \eta + \frac{1}{\eta}\right) q^3 + \cdots \ . \end{aligned}$$

Taking $\eta \to 1$, SCI yields 1. This would be reproduced from twisted index with $\mathfrak{g} = 0$.

Why does $\mathcal{I}^{\text{sci}}=1$ implies TQFT?

SCI in Degenerate Limits $\nu=\pm 1$ = Hilbert Series on Coulomb/Higgs Branches

Due to this and definition of rank-0 theory, SCI or Hilbert series on Coulomb/Higgs branch only count identity operator.

TQFT has NO local operators

Hilbert-series counts gauge invariant local operators. TQFT has no local operators except the identity operator.

Modular Data from Bethe-Vacua

S_b^3 Partition Function from Localization Formula

From the Localization Recipes in previous slides with $k=-3/2$, $R_\Phi=\Delta_\Phi=1/2$, charge $+1$, one can construct $\mathcal{Z}^{S_b^3}$ as

$$\mathcal{Z}^{S_b^3}_{\mathcal{T}_{\rm min}}(b,m,\nu)=\int\frac{dZ}{\sqrt{2\pi\hbar}}\,e^{-\frac{Z^2+2Z\left(m+(i\pi+\frac{\hbar}{2})\nu\right)}{2\hbar}}\psi_\hbar(Z)$$

Bethe-Vacua

$$\text{Bethe-vacua of }\mathcal{T}_{\rm min}\;:\;\left\{z\;:\;\frac{(z-1)e^{-m-i\pi\nu}}{z^2}=1\right\}$$

Modular Data from Bethe-Vacua

Handle-Gluing and Fibering Operators

$$\left\{\mathcal{F}_\alpha(m=0,\nu\to\pm1,s=-1)\right\}_{\alpha=0,1} \longrightarrow \left\{\exp\left(-\frac{7i\pi}{60}\right),\exp\left(\frac{17i\pi}{60}\right)\right\}$$

$$\left\{\mathcal{H}_\alpha(m=0,\nu\to\pm1,s=-1)\right\}_{\alpha=0,1} \longrightarrow \left\{\frac{5-\sqrt{5}}{2},\frac{5+\sqrt{5}}{2}\right\}$$

Modular Matrices (S, T)

$$S=\begin{pmatrix}\sqrt{\frac{1}{10}\left(\sqrt{5}+5\right)} & -\sqrt{\frac{1}{10}\left(5-\sqrt{5}\right)}\\ -\sqrt{\frac{1}{10}\left(5-\sqrt{5}\right)} & -\sqrt{\frac{1}{10}\left(\sqrt{5}+5\right)}\end{pmatrix}, \qquad T=\begin{pmatrix}1 & 0\\ 0 & \exp(-\frac{2\pi i}{5})\end{pmatrix}$$

Anyons of Lee-Yang TQFT from Loop Operators

Identification of Simple Objects

For a $U(1)$ gauge theory, the supersymmetric dyonic loop operator $\mathcal{O}_{(p,q)}$ of (electric charge, magnetic charge)=(p,q) is

$$\mathcal{O}_{(p,q)} = z^p(1-z^{-1})^q \ .$$

Two simple objects in Lee-Yang MTC identified as a supersymmetric loop operators as

$$\mathcal{O}_{\alpha=0} = \mathcal{O}_{(0,0)} = (\text{identity operator}) \ , \quad \mathcal{O}_{\alpha=1} = \mathcal{O}_{(p,q)=(1,0)} \ .$$

Completing S-Matrix

Completing S-matrix other than the first row/column, the belows are required to compute.

$$W_{\beta=0,1}(0) = 1 \ , \quad W_{\beta=0}(1) = z_0 = \frac{1}{2}(\sqrt{5}-1) \ , \quad W_{\beta=1}(1) = z_1 = \frac{1}{2}(-\sqrt{5}-1)$$

Infinitely Many Examples

Table of Examples [Gang, Kim, Lee, MS, Yamazaki 2021]

$\mathcal{T}_{\text{rank }0}$	$\text{TFT}_\pm[\mathcal{T}_{\text{rank }0}]$	Set of $\{\lvert S^\pm_{0\alpha}\rvert\}$	$\exp(-F)$
$\mathcal{T}_{\min}$	(Lee-Yang)	$\{\sqrt{\frac{5+\sqrt{5}}{10}}, \sqrt{\frac{5-\sqrt{5}}{10}}\}$	$\sqrt{\frac{5-\sqrt{5}}{10}}$
$(U(1)_1+H)$	$\text{Gal}_d(SU(2)_6)/\mathbb{Z}_2^f$ (with $d=\zeta_6^3$)	$\{2\zeta_6^1, 2\zeta_6^3\}$	$2\zeta_6^1$
$SU(2)_k^{\frac{1}{2}\oplus\frac{1}{2}}$ ($\lvert k\rvert>1$)	$\text{Gal}_d(SU(2)_{4\lvert k\rvert-2})/\mathbb{Z}_2^f$ (with $d=\zeta_{4\lvert k\rvert-2}^{2\lvert k\rvert-1}$)	$\{2\zeta_{4\lvert k\rvert-2}^{2n-1}\}_{n=1}^{\lvert k\rvert}$	$2\zeta^1_{4\lvert k\rvert-2}$
$T[SU(2)]_{k_1,k_2}$	See the caption	$\{(\frac{1}{\sqrt{2}}\zeta^n_{\lvert k_1k_2-1\rvert-2})^{\otimes 2}\}_{n=1}^{\lvert k_1k_2-1\rvert-1}$	$\frac{1}{\sqrt{2}}\zeta^1_{\lvert k_1k_2-1\rvert-2}$
$\frac{T[SU(2)]}{SU(2)^{\text{diag}}_{\lvert k\rvert=3}}$	$(\text{Lee-Yang})^{\otimes 2}\otimes U(1)_2$	$\{\frac{1}{\sqrt{10}}^{\otimes 4}, \frac{5+\sqrt{5}}{10\sqrt{2}}^{\otimes 2}, \frac{5-\sqrt{5}}{10\sqrt{2}}^{\otimes 2}\}$	$\frac{5-\sqrt{5}}{10\sqrt{2}}$
$\frac{T[SU(2)]}{SU(2)^{\text{diag}}_{\lvert k\rvert=4}}$	$\frac{\text{Gal}_{\zeta_{10}^7}(SU(2)_{10})\times SU(2)_2}{\mathbb{Z}_2^{\text{diag}}}$	$\{\frac{1}{2}, \frac{1}{2\sqrt{3}}^{\otimes 5}, \frac{3+\sqrt{3}}{12}^{\otimes 2}, \frac{3-\sqrt{3}}{12}^{\otimes 2}\}$	$\frac{3-\sqrt{3}}{12}$
$\frac{T[SU(2)]}{SU(2)^{\text{diag}}_{\lvert k\rvert=5}}$	$\text{Gal}_d((G_2)_3)\otimes U(1)_{-2}$ $\left(d=\sqrt{\frac{5}{84}+\frac{1}{4\sqrt{21}}}\right)$	$\{\frac{1}{\sqrt{6}}^{\otimes 2}, \frac{1}{\sqrt{14}}^{\otimes 6}, \sqrt{\frac{5}{84}\pm\frac{1}{4\sqrt{21}}}^{\otimes 2}\}$	$\sqrt{\frac{5}{84}-\frac{1}{4\sqrt{21}}}$
$\frac{T[SU(2)]}{SU(2)^{\text{diag}}_{\lvert k\rvert\geq 6}}$	?	$\{\frac{1}{\sqrt{2\lvert k\rvert-4}}^{\otimes(\lvert k\rvert-3)}, \frac{1}{\sqrt{2\lvert k\rvert+4}}^{\otimes(\lvert k\rvert+1)}, (\frac{1}{\sqrt{8\lvert k\rvert-16}}+\frac{1}{\sqrt{8\lvert k\rvert+16}})^{\otimes 2}, (\frac{1}{\sqrt{8\lvert k\rvert-16}}-\frac{1}{\sqrt{8\lvert k\rvert+16}})^{\otimes 2}\}$	$\frac{1}{\sqrt{8\lvert k\rvert-16}}-\frac{1}{\sqrt{8\lvert k\rvert+16}}$

Natural Questions

Classification of Rank-0 SCFTs and Dualities

Does the same TQFT imply dualities between rank-0 SCFTs?

Relation with Rozansky-Witten Theory

In RW theories, one can obtained Unitary TQFTs. We conjectures our Non-Unitary TQFTs are Galois Conjugate of them.

Relation with 4d $\mathcal{N} = 2$ Argyres-Douglas Theories

Thank You

THANK YOU for YOUR ATTENTION

ありがとうございました