S-matrix Unitarity toward UV Completion

-Relation to renormalizability-

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Introduction of myself

Gravity

cosmology | **General relativity** | **Quantum theory**

Geometry

CMB
Large scale structure
Causal structures

Black Holes
- Theorems
- Exact solutions
- Large dimensional limit

Scattering process
Causal structures

QFT on curved spacetime

Higher Dimensions

Motivation (Quantum gravity)

General Relativity

Proposed by Einstein in 1916

Tested by many observations

Light bending by sun
Perihelion shift of Mercury
Cavendish's experiment
Cosmology

Recent observation: Gravitational waves

10^{-21} order precision

$$\frac{\text{Size of Hydrogen atom}}{\text{Distance between Sun and Earth}}$$

The world record in any experiment

Motivation (Quantum gravity)

GR as low energy gravitational theory

GR:

observationally good theory

⇕

singularity theorem ← Initial singularity in Universe
Singularities in black hole

Quantization may resolve singularities in GR

UV problems of quantization of GR

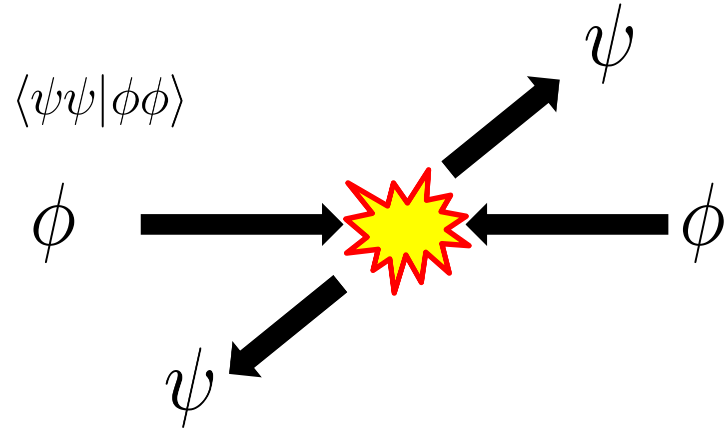
Violation of Unitarity in UV scattering

Not renormalizable

Understanding unitarity and renormalizability is important!!

I explain definitions of them from next slide

Feynman diagram



Technique to calculate scattering amplitude

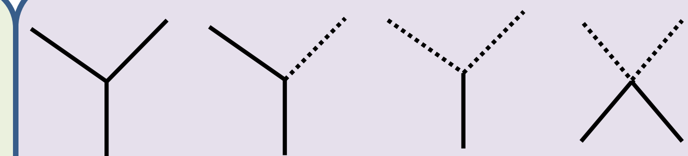
Partition function $Z = \int \mathcal{D}\phi(x)\exp(iS)$

Example $\dfrac{\int \mathcal{D}\phi(x)\exp(iS)\psi_\infty\psi_\infty\phi_{-\infty}\phi_{-\infty}}{\int \mathcal{D}\phi(x)\exp(iS)}$

$$S = \int d^4x \tfrac{1}{2}[-(\nabla\phi)^2 - m^2\phi^2] + \tfrac{1}{2}[-(\nabla\psi)^2 - M^2\psi^2] + \lambda_1\phi^3 + \lambda_2\phi^2\psi + \lambda_3\phi\psi^2 + \lambda_4\phi^2\psi^2 + \cdots$$

$(k^2+m^2)^{-1}$ $(k^2+M^2)^{-1}$

propagator

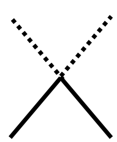


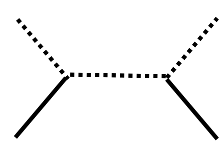
λ_1 λ_2 λ_3 λ_4

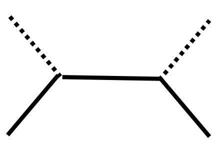
vertex

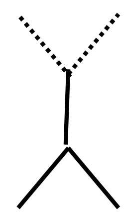
Scattering amplitude $\langle\psi\psi|\phi\phi\rangle$

time

 λ_4

 $\lambda_3 \frac{1}{k^2+M^2}\lambda_3$

 $\lambda_2 \frac{1}{k^2+m^2}\lambda_2$

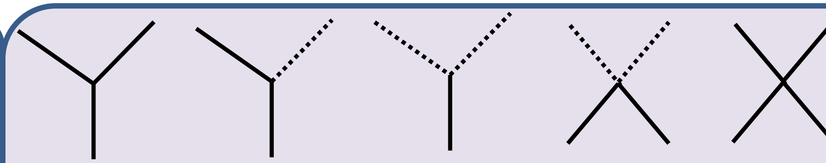
 $\lambda_1 \frac{1}{k^2+m^2}\lambda_3$

...... etc.

Feynman diagram

$(k^2+m^2)^{-1}$ $\quad$ $(k^2+M^2)^{-1}$

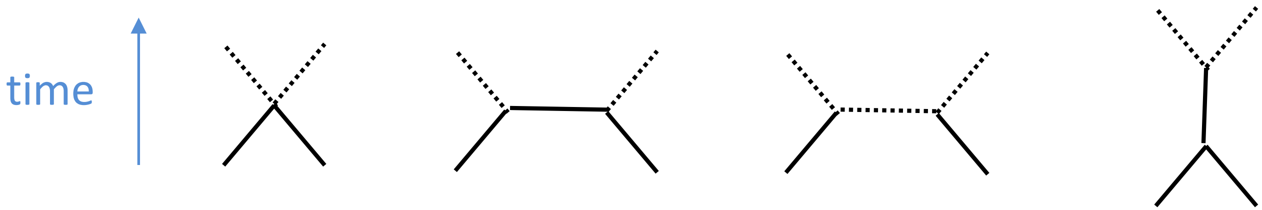
propagator

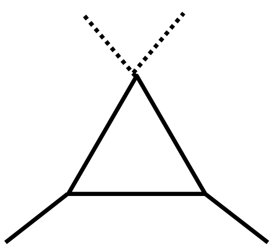


λ_1 $\quad$ λ_2 $\quad$ λ_3 $\quad$ λ_4 $\quad$ λ_5

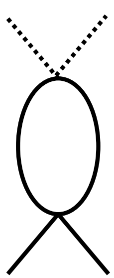
vertex

Scattering amplitude $\quad \langle\psi\psi|\phi\phi\rangle$





$$\int d^4k \;\; \lambda_4 \frac{1}{k^2+m^2} \lambda_1 \frac{1}{k^2+m^2} \lambda_1 \frac{1}{k^2+m^2}$$



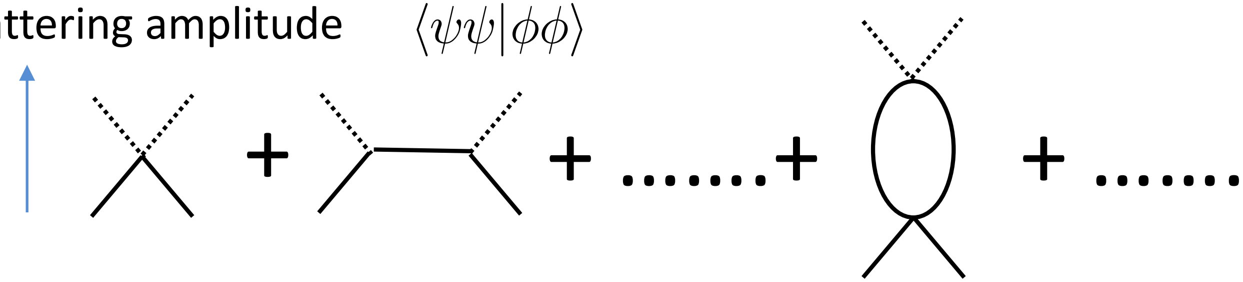
$$\int d^4k \;\; \lambda_4 \frac{1}{k^2+m^2} \lambda_5 \frac{1}{k^2+m^2} \to \infty$$

...... etc.

Renormalizability

Scattering amplitude $\langle\psi\psi|\phi\phi\rangle$

time

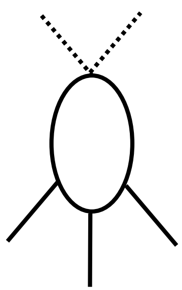
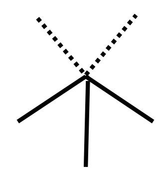


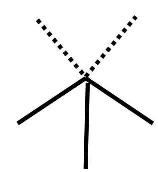
$$\lambda_4 + \ldots\ldots\ldots + \int d^4k\ \lambda_4\frac{1}{k^2+m^2}\lambda_5\frac{1}{k^2+m^2} \to \cancel{\infty} + \ldots\ldots\ldots$$

$$\lambda_4 = \tilde{\lambda}_4 - \cancel{\infty}$$

Get finite value

Renormalization

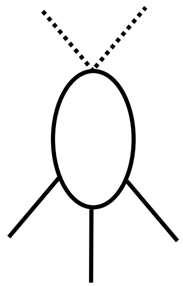
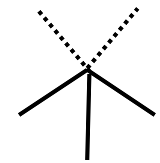
If  diverge but not have  ,

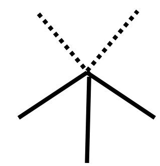
to renormalize we need to add 

in the original theory

Renormalizability

$$S = \int d^4x \frac{1}{2}[-(\nabla\phi)^2 - m^2\phi^2] + \frac{1}{2}[-(\nabla\psi)^2 - M^2\psi^2] + \lambda_1\phi^3 + \lambda_2\phi^2\psi + \lambda_3\phi\psi^2 + \lambda_4\phi^2\psi^2 + \cdots$$

If  diverge but not have  ,

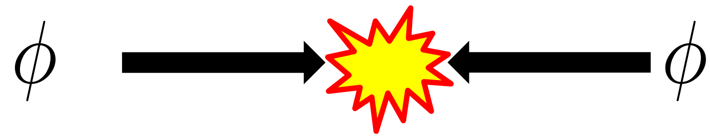
to renormalize we need to add 

in the original theory $\lambda_5\phi^3\psi^2$

If we need to add infinite vertices,
theory is not renormalizable

Such a theory has infinite parameters
but we can fix some of them by observation.
Thus, the theory can not give predictions.

Unitarity



Scattering amplitude

$$\langle\psi\psi|\phi\phi\rangle \quad \langle\psi\psi\psi|\phi\phi\rangle \quad \langle\psi\psi\phi|\phi\phi\rangle \quad \cdots\cdots\cdots\cdots$$

Probability

$$|\langle\psi\psi|\phi\phi\rangle|^2 \quad |\langle\psi\psi\psi|\phi\phi\rangle|^2 \quad |\langle\psi\psi\phi|\phi\phi\rangle|^2 \quad \cdots\cdots\cdots\cdots$$

Sum of all probability gives 1 (that is 100 %)

$$1 = \sum |\langle\Psi_f|\phi\phi\rangle|^2 = \sum \langle\Psi_f|\phi\phi\rangle^* \langle\Psi_f|\phi\phi\rangle = \sum \langle\phi\phi|\Psi_f\rangle\langle\Psi_f|\phi\phi\rangle$$

(Sum is taken for all possible final state)

For any initial state, we have the same equation

$$1 = \sum \langle\Psi_i|\Psi_f\rangle\langle\Psi_f|\Psi_i\rangle \quad \Longrightarrow \quad \boxed{\mathbf{1} = S^\dagger S}$$

unitarity

$$S := \langle\Psi_f|\Psi_i\rangle$$

Unitarity and renormalizability

Conjecture by Llwellyn Smith (1974)

If no physical ghost propagation （no negative norm）

High energy limit of **Tree-Unitarity = Renormalizability**

(Tree-level approximation of unitarity)

Y-M theory	Yes	Yes
W-S model	Yes	Yes
Massive vector	No	No
4-Fermi	No	No
Einstein gravity (Genaral Reratitivy)	No	No

No counterexample!

UV-unitarity vs Renormalizability

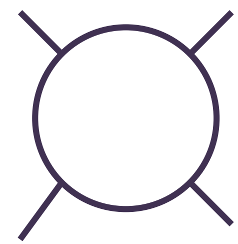
Tree-unitarity is simpler than renormalizability.
Tree-unitarity is a good tool to investigate perturbative UV completion.

Tree-unitarity

$$V\frac{1}{\omega^2-p^2}V$$

renormalizability

$$\int d\omega d^d k \;\; V\frac{1}{\omega^2-p^2}V\ldots\ldots$$



Recently, bottom-up approach to QG is discussed based on unitarity of S-matrix.
→ Such theories are expected to be automatically renormalizable.

Theories that we analyze

- theory without Lorentz symmetry (Lifshitz scaling scalar theory)

  Horava Lifshitz gravity

  Checking tree-unitarity may be easier than checking renormalizability.

  Check whether the relation is purely quantum origin or not.

  Less symmetric theory is better,
  because symmetry may do something.

  Why not break the Lorentz symmetry?

- Higher order derivative scalar theory (with Lorentz symmetry)

  $R_{\mu\nu}^2$ model

  Ghost modes exist but this model is renormalizable.

  "unitairy" is violated

  Counterexample of the relation between unitarity & renormalizability?

  → "Unitarity" here means "S-matrix unitarity".
  Then the relation holds even in the theory with ghost mode.

Contents

[…] means dimension of …

Power-counting theorem

It is generally said

If mass (or momentum) dimension of coupling constant is non-negative, the coupling term is renormalizable.

First of all, we fix the dimension of field from the second order action.

$$S_2 = \int d^d x \ \frac{1}{2}[-(\nabla\phi)^2 - m^2\phi^2]$$

$$[p] = 1, \ \ [E] = 1, \ \ [dt] = [dx] = -1 \qquad [\phi] = (d-2)/2$$

$$[\nabla] = 1$$

$$0 = [S_2] = d[dx] + 2[\nabla] + 2[\phi]$$

Then, we can calculate dimension of a coupling term

$$S_{int} = \lambda \underline{\int d^d x} \ \underline{(\nabla^{a_1}\phi)}\ldots\underline{(\nabla^{a_n}\phi)}$$

$$0 = [S_{int}] = [\lambda] + \underline{d[dx]} + \underline{a_1[\nabla] + [\phi]} + \ldots + \underline{a_n[\nabla] + [\phi]}$$

$$[\lambda] = -d[dx] - a_1[\nabla] - [\phi] - \ldots - a_n[\nabla] - [\phi] \ \ \geq 0$$

Condition for renormalizability

[...] means dimension of ...

Power-counting theorem

It is generally said

If mass (or momentum) dimension of coupling constant is non-negative,
~~the coupling term is renormalizable.~~

the mass dimension of coupling constant for any counter term is non-negative

Ex. $[\phi] = 1 (> 0)$

$$S_{int} = \lambda_n \int d^4x \phi^n \qquad [\lambda_n] \geq 0, \quad (n \leq 4) \qquad [\lambda_n] < 0, \quad (n \geq 5)$$

$$S_{int} = \bar{\lambda}_n \int d^4x \phi^n (\partial_\mu \phi)^2 \qquad [\bar{\lambda}_n] \geq 0, \quad (n = 4) \qquad [\bar{\lambda}_n] < 0, \quad (n \geq 1)$$

Ex. $[\phi] = 0$

$$S_{int} = \lambda_n \int d^4x \phi^n \qquad [\lambda_n] = 4 \geq 0, \quad (\text{ for } \quad \text{any} \ \ n)$$

Even if all dimensions of coupling constants are non-negative,
the infinite number of counter terms are required.

NOT Renormalizable!!

Extended Power-counting theorem

$$S_{int} = \lambda \int d^dx \, \overbrace{(\partial_x^{a_1}\phi)\ldots(\partial_x^{a_n}\phi)\underbrace{(\partial_x^{b_1}\phi)\ldots(\partial_x^{b_m}\phi)}_{\text{Dimension of any combination of interaction term } < d}}^{\text{Dimension of whole interaction term } \leqq d}$$

Example $[\phi] = -1$ in 4 –dim (d=4) $\quad [\partial_x \phi] = 0 \quad [\partial_x^5 \phi] = 4$

$$S_{int} = \lambda \int d^4x \, \overbrace{(\partial_x\phi)\underbrace{(\partial_x\phi)(\partial_x\phi)(\partial_x^5\phi)}}$$

$$[\lambda] = 0$$

$$[\overbrace{(\partial_x\phi)(\partial_x\phi)(\partial_x\phi)(\partial_x^5\phi)}] = 4 \leq 4$$

$$[\underbrace{(\partial_x\phi)(\partial_x\phi)(\partial_x^5\phi)}] = 4 < 4$$

NOT satisfied

NOT Renormalizable!!

Extended Power-counting theorem

Dimension of whole interaction term ≦d

$$S_{int} = \lambda \int d^d x \overbrace{(\partial_x^{a_1}\phi)\ldots(\partial_x^{a_n}\phi)\underbrace{(\partial_x^{b_1}\phi)\ldots(\partial_x^{b_m}\phi)}}$$

Dimension of any combination of interaction term <d

Example $[\phi] = -1$ in 4 –dim (d=4) $\quad [\partial_x^2\phi] = 2$

$$S_{int} = \lambda \int d^4 x \overbrace{(\partial_x^2\phi)\underbrace{(\partial_x^2\phi)(\partial_x^2\phi)(\partial_x^2\phi)}}$$

$$[\lambda] = 0$$

$$[\overbrace{(\partial_x^2\phi)(\partial_x^2\phi)(\partial_x^2\phi)(\partial_x^2\phi)}] = 4 \leq 4$$

$$[\underbrace{(\partial_x^2\phi)(\partial_x^2\phi)(\partial_x^2\phi)}] = 3 < 4$$

Renormalizable!!

Why traditional Power-counting works?

$$S_2 = \int d^d x \ \frac{1}{2}[-(\nabla\phi)^2 - m^2\phi^2]$$

Dimension of ϕ

$$0 = d[dx] + 2[\partial_x] + 2[\phi] = -d + 2 + 2[\phi]$$

$[\phi] = \frac{d-2}{2} > 0$ for $d \geq 3$

$$[(\partial_x^a \phi)] > 0$$

$$S_{int} = \lambda \int d^d x \overbrace{(\partial_x^{a_1}\phi)\ldots(\partial_x^{a_n}\phi)\underbrace{(\partial_x^{b_1}\phi)\ldots(\partial_x^{b_m}\phi)}_{\text{Dimension of any combination of interaction term } <d}}^{\text{Dimension of whole interaction term } \leqq d}$$

However, if ① Lorentz inv. does not hold or ② Higher derivatives are involved,

$[\phi] \leq 0$ is achieved.

Contents

Unitarity bound

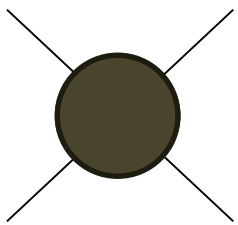
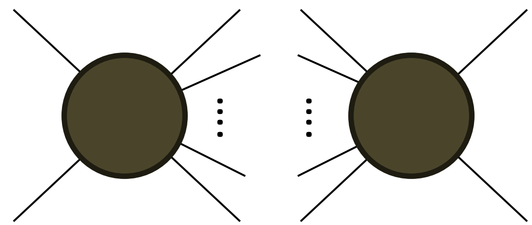
We use unitarity bound to check the unitarity of coupling term.

What is unitarity bound?

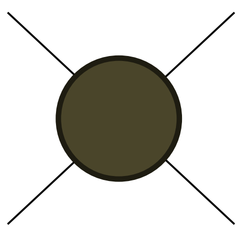
(S-matrix) unitarity

$$S^\dagger S = 1 \qquad\qquad 1 = \sum \langle \Psi_i | \Psi_f \rangle \langle \Psi_f | \Psi_i \rangle$$

Optical theorem

Im  $=$ $\sum$ 

Unitarity bound

 $\leqq$ const.

Unitarity and renormalizability

Conjecture by Llwellyn Smith (1974)

If no physical ghost propagation （no negative norm）

High energy limit of **Tree-Unitarity = Renormalizability**

(Tree-level approximation of unitarity)

Y-M theory	Yes	Yes
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No counterexample!

Contents

Theory w/o Lorentz sym. (Lifshitz scaling theory)

Theory does not have Lorentz symmetry.

→ no symmetry between time and spacial derivatives

Positive integer

We may consider $S_2 = \int dt d^d x \ \phi[-\partial_t^2 - \Delta^z]\phi$

$[\partial_t] = z[\partial_x]$

$[p] = 1, \ [E] = z \qquad [dt] = -z, [dx] = -1$

$$0 = [S_2] = [dt] + d[dx] + 2[\partial_t] + 2[\phi] \Rightarrow [\phi] = (d-z)/2$$

$S_4 = \int dt d^d x (\partial_x^{a1}\phi)(\partial_x^{a2}\phi)(\partial_x^{a3}\phi)(\partial_x^{a4}\phi)$

$(a_1 \le a_2 \le a_3 \le a_4)$

Condition for renormalizability in quartic interaction term

$a_1 + a_2 + a_3 + a_4 \le 3z - d$

$a_2 + a_3 + a_4 \le (5z - d - 1)/2$

$a_3 + a_4 \le 2z - 1$

Two-particle scattering

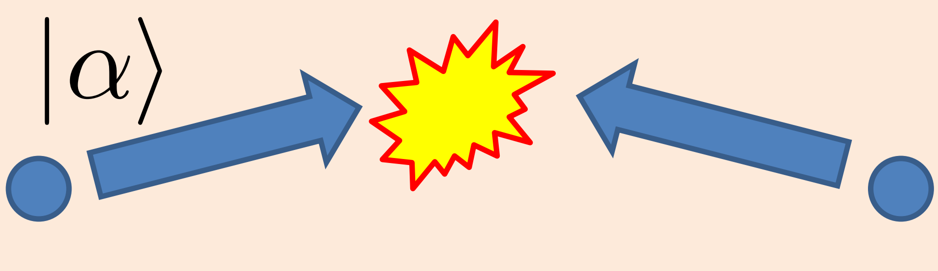
High energy limit

Relativistic theory

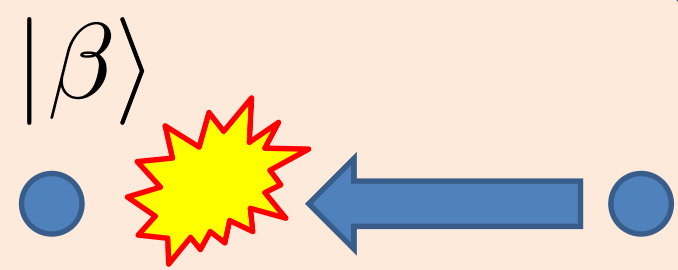
We can always take CoM frame.
High energy limit => ($E \to \infty$, $P=0$)

Lifshitz scaling theory

No Lorentz symmetry.
High energy limit with non-zero P (which can be diverge) can give different conditions.



Energy of both particles go to infinity. (Including CoM)



Only one goes to infinity

Unitarity Bound

$$S_2 = \int dt d^d x \ \phi[-\partial_t^2 - (-\Delta - \cdots - \Delta^z)]\phi$$

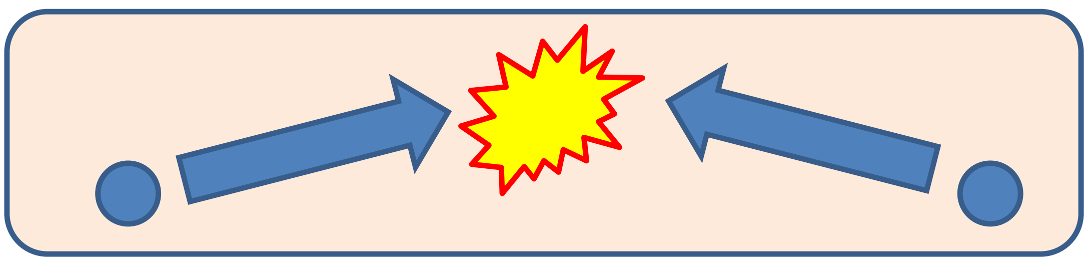
$$S_4 = \int dt d^d x (\partial_x^{a_1}\phi)(\partial_x^{a_2}\phi)(\partial_x^{a_3}\phi)(\partial_x^{a_4}\phi) \quad (a_1 \leq a_2 \leq a_3 \leq a_4)$$

$$\mathcal{M}(\mathbf{p}_1, \mathbf{p}_2 \to \mathbf{k}_1, \mathbf{k}_2) = p_1^{a_1} p_2^{a_2} k_1^{a_3} k_2^{a_4} + [\text{ perm. }]$$

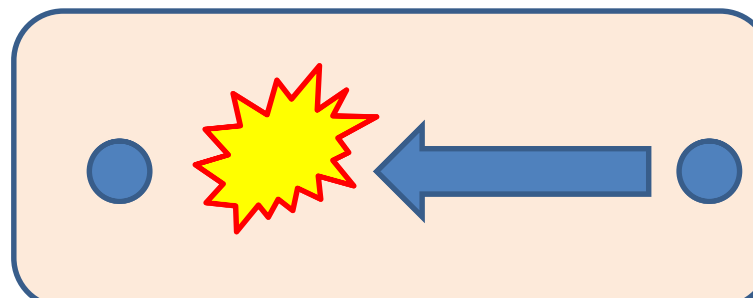
$$\mathcal{M}(\alpha \to \alpha) \qquad P^{a_1+a_2+a_3+a_4} \qquad \boxed{a_1 + a_2 + a_3 + a_4 \leq 3z - d}$$

$$\mathcal{M}(\beta \to \beta) \qquad P^{a_3+a_4} \qquad \boxed{a_3 + a_4 \leq 2z - 1}$$

$$\mathcal{M}(\alpha \to \beta) \qquad P^{a_2+a_3+a_4} \qquad \boxed{a_2 + a_3 + a_4 \leq (5z - d - 1)/2}$$



$|\alpha\rangle$



$|\beta\rangle$

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No Lorentz inv.

coincident

Contents

Unitarity and renormalizability

Counterexample?

$R_{\mu\nu}^2$ model

Negative norm modes exist but this model is renormalizable.

"unitairy" is violated

What does unitarity mean?

Unitarity (in physics) means

1, S-matrix unitarity

$$SS^\dagger = 1$$

2, norm positivity

$$\langle\psi|\psi\rangle > 0 \quad \text{for any } |\psi\rangle$$

This is related to the renormalizability!

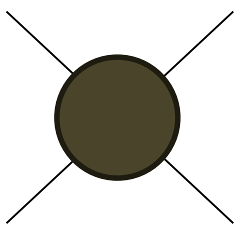
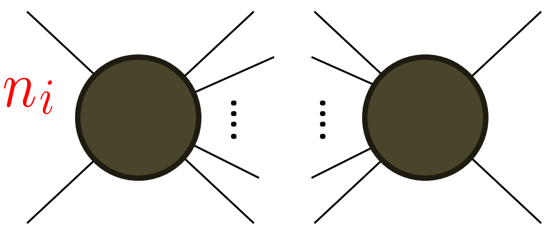
S-matrix unitarity and unitarity bound

What happens if negative norms exist

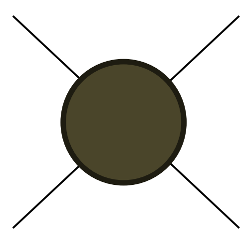
(S-matrix) unitarity

$$SS^\dagger = 1$$

Optical theorem

Im  $= \sum (-1)^{n_i}$ 

Unitarity bound

const. $\geqq$ 

S-matrix unitarity and unitarity bound

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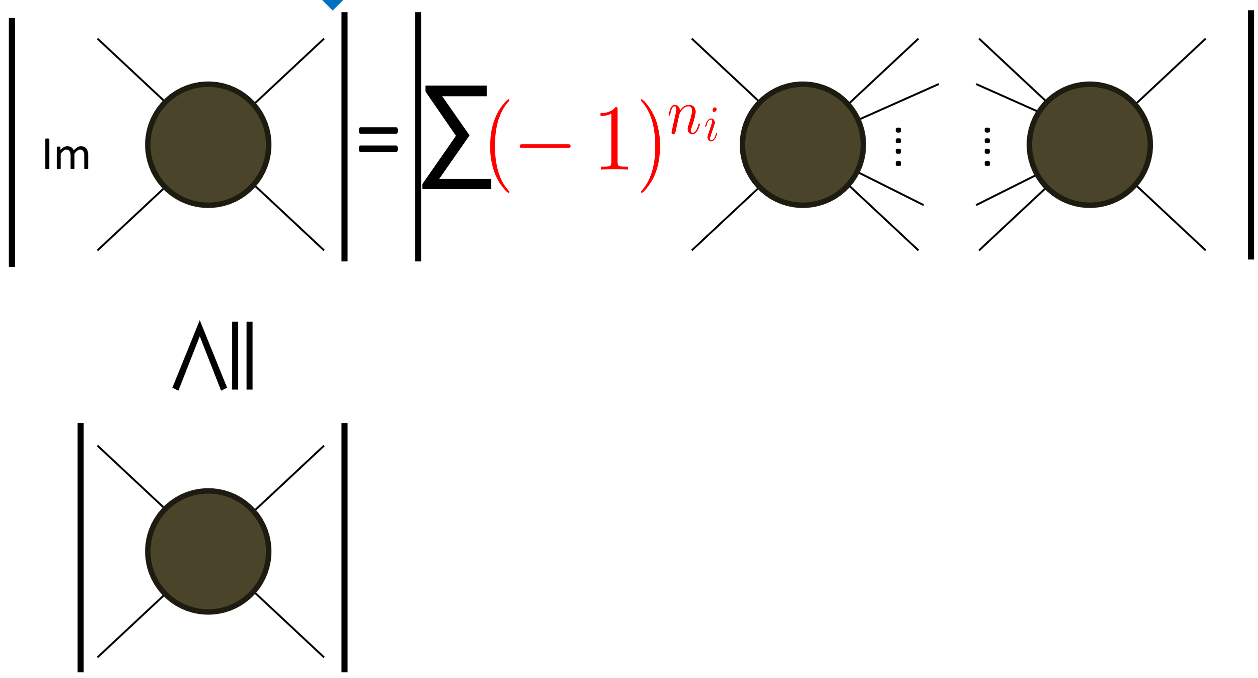


Scalar with Higher order derivative

- Kinetic term with higher order derivative

$$S_{2\phi} = \int dt d^3x \ \ \phi(\Box - m_1^2)(\Box - m_2^2)\phi$$

$$[\phi] = 0$$

$$\phi = \sqrt{2/(m_2^2 - m_1^2)}\,(\psi_1 - \psi_2)$$

$$S_{2\phi} = \int dt d^3x \ \ \psi_1(\Box - m_1^2)\psi_1 - \underline{\psi_2(\Box - m_2^2)\psi_2}$$

Negative Norm

Renormalizability $\Longleftrightarrow$ S-matrix Unitarity

Coincident!! $SS^\dagger = 1$

(norm positivity is not important!)

Quadratic Gravity ($R_{\mu\nu}{}^2$ Gravity)

Action

$$S_{gravity} = \int d^4x\sqrt{-g}\left(\Lambda + \frac{1}{\kappa^2}R + \alpha R^2 + \beta R_{\mu\nu}^2\right),$$

$$S_{matter} = \int d^4x\sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4 + \xi\phi^2 R\right).$$

- Renormalizable (if $\beta \neq 0$), (Stelle 1977)
- Negative norm state due to higher order derivative

DoFs of graviton

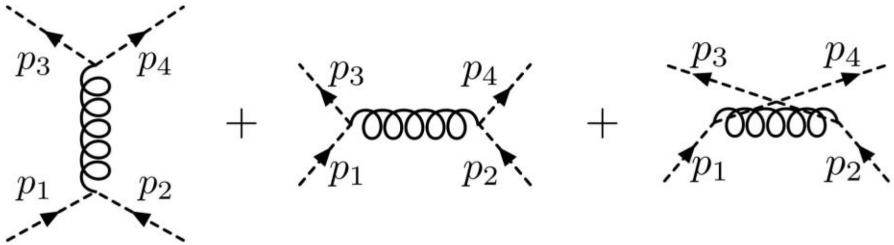
2 massless spin-2 DoFs
5 massive spin-2 Dofs ← Negative norm states (appearing only for $\beta \neq 0$)
1 massive scalar Dofs

with scalar field ϕ

Scalar-Field Scattering

Action
$$S_{gravity} = \int d^4x\sqrt{-g}\left(\Lambda + \frac{1}{\kappa^2}R + \alpha R^2 + \beta R_{\mu\nu}^2\right),$$
$$S_{matter} = \int d^4x\sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4 + \xi\phi^2 R\right).$$

Scattering via graviton exchange



$$M = -\lambda + \frac{2}{\beta}g(\cos\theta) - \frac{3(2\alpha+\beta)}{2\beta(3\alpha+\beta)}$$

Unitarity bound is satisfied,
→ S-matrix Unitarity is satisfied

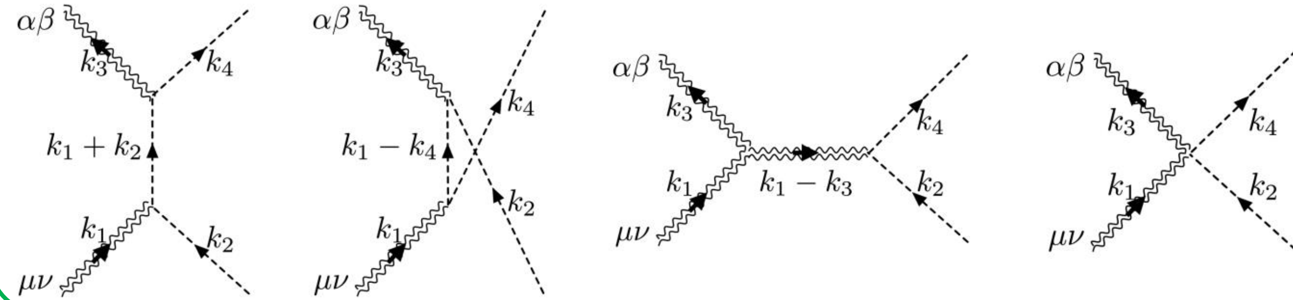
cf: for β=0, scattering amplitude becomes

$$M = 2\kappa^2 s f(\cos\theta) + \mathcal{O}(s^0) = \mathcal{O}(s^1)$$

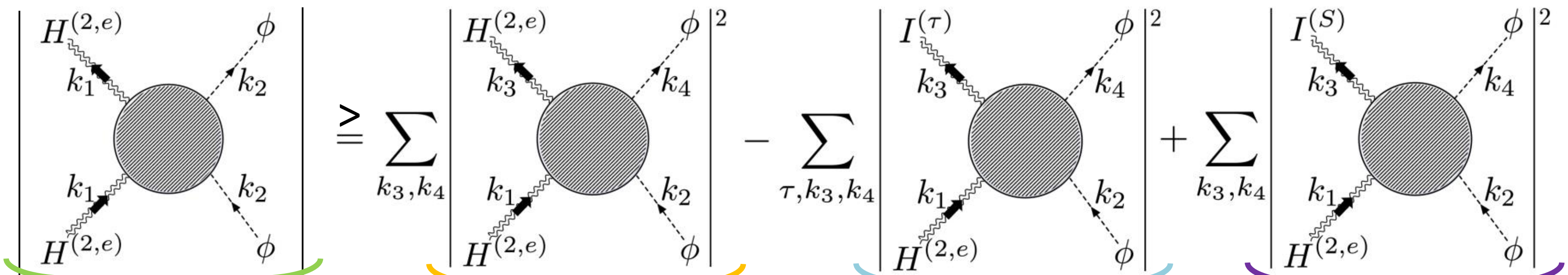
No S-matrix unitarity

Consistent with the condition on renormalizability

Graviton-Scalar Scattering



S-matrix unitarity



$$\left|\mathcal{A}\left(H^{(2,e)}+\phi\to H^{(2,e)}+\phi\right)\right|$$

~k^2

$$\left|\mathcal{A}\left(H^{(2,e)}+\phi\to H^{(2,e)}+\phi\right)\right|^2$$

~k^4

$$-\left|\mathcal{A}\left(H^{(2,e)}+\phi\to I^{(2,e)}+\phi\right)\right|^2$$

~k^4

$$-\left|\mathcal{A}\left(H^{(2,e)}+\phi\to I^{(1,e)}+\phi\right)\right|^2$$

$$-\left|\mathcal{A}\left(H^{(2,e)}+\phi\to I^{(0,e)}+\phi\right)\right|^2$$

$$+\left|\mathcal{A}\left(H^{(2,e)}+\phi\to I^{(S)}+\phi\right)\right|^2$$

$H^{(2.e)}$: massless positive-norm spin2

$I^{(i.e)}$: massive negative-norm spin2

Summary

Unitarity ⟺ Renormalizability

- S-matrix Unitarity
- ~~norm positivity~~

Under the norm positivity condition, the relation is reduced to that between renormalizability and unitarity.

Discussion

Quadratic gravity ($R_{\mu\nu}^2$ gravity) as Quantum gravity

- Renormalizable
- S-matrix Unitarity is satisfied. (Perturbative UV completion would work.)

How to interpret the negative norm (negative possibility) ??

Negative norm gravitons do not appear in the asymptotic state.
→ There would be a consistent formalization
with unstable negative-norm state.

Thank you